Exercises: Micro-scale flows

Charles Baroud

Sofia, April 2011

Below are four exercises that go back on what was seen in the lecture. The calculations in each of them are simple. Try to do them with minimal math!

1 Which flow is a low Re flow?

Observe the two movies marked Taylor 1 and Taylor 2.

- Which one of the situations corresponds to a low Re flow? How do you know it? There are severeal indeces.
- What is the main fluid property that varies between the two movies?
- In the high Reynolds number case: Estimate the viscous diffusion time from the formula, suposing that the liquid is water. How does it compare with the time that you would extract from the movie? Why?

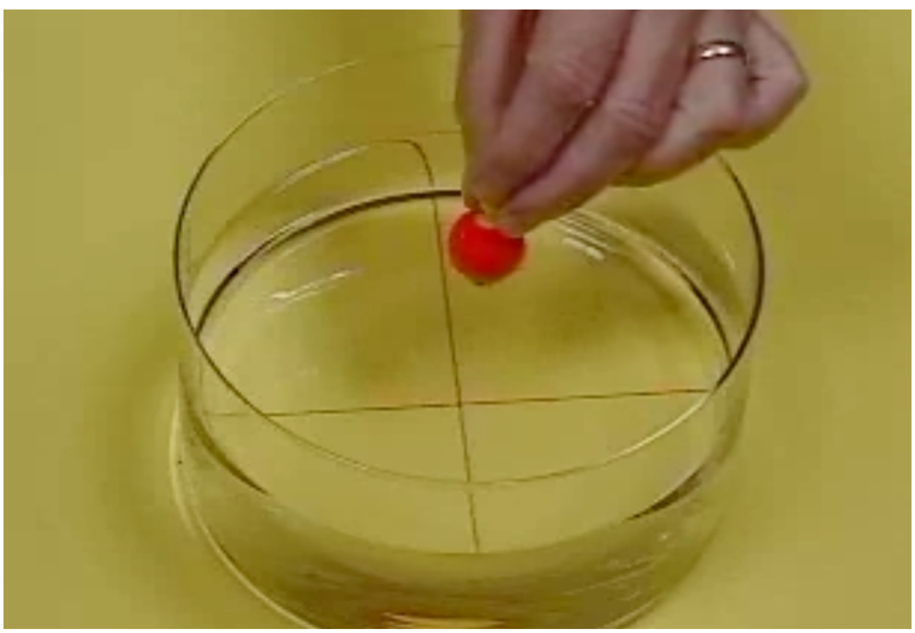


Fig. 1- Screen shot from movies.

1.1 Answer

The movie Taylor 2 is the low Re case. You can see this through different aspects of the fluid:

1. In the movie Taylor 1, we see waves on the surface of the liquid. The viscosity of the second case dissipates the waves very quickly and so we do not see them.
2. In the movie Taylor 2, we see that it takes a very long time for the liquid, which had wetted the cork, to drain. This is a sign that the viscous effects are slowing down the draining
3. Finally, the liquid reacts almost immediately to the moving bowl in the second case, a sign that the viscous diffusion of momentum is very quick.

The main difference between the two movies is the viscosity.
The diffusion time in the low Re case is almost zero.
In the high Re case, I would have estimated that $\tau_{diff} = L^2/\nu$, with $L \simeq 3$ cm and $\mu = 0.01$ cm^2/s. This would give $\tau_{diff} \simeq 1000$ s, which is much longer than what we observe!
This means that the momentum is being transfered by a different mechanism than just viscous diffusion alone. We can imagine that there are vortices that are formed which drive the motion of the cork, for example. Probably what makes the time much shorter is that the bowl is rotating and not just translating... This leads to a phenomenon called "Ekman Pumping" but it is too complicated for this course.

2 Hydrodynamic resistance

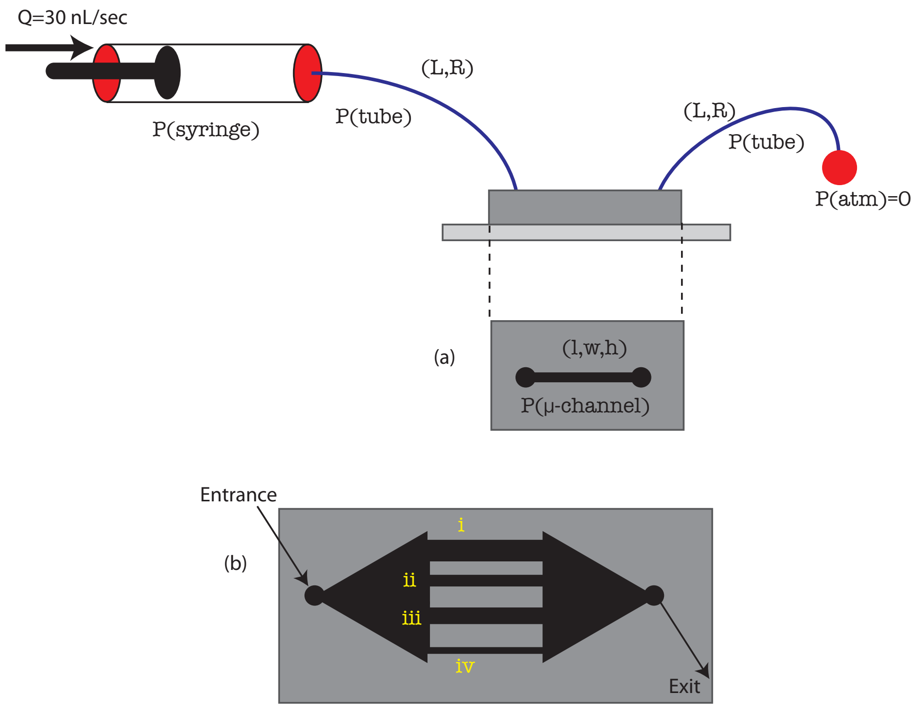


Fig. 2- Two microfluidic circuits.

1. Consider the microfluidic circuit in Fig. 1(a). Suppose that the liquid that is pushed is water ($\mu = 10^{-3}$ Pa.s). What is the highest Reynolds number in this circuit? Can we talk about Stokes flows everywhere in the system? The dimensions of each element are the following:

Element	Width or Radius	Length
Syringe	2 mm	4 cm
Tube ($\times 2$)	0.5 mm	20 cm
Microchannel	width=100 μm	2 cm
	height=20 μm	

2. Calculate the pressure drop in each one of the elements of the circuit. Which element provides the dominant resistance? What would the flow rate be if the syringe pump is replaced with a pressure source that applies $P = 10^4$ Pa?

3. Now replace the single microfluidic channel by the network shown in part (b). The heights of all channels are all the same ($h = 20$ μm). The widths are: (i) 400 μm. (ii) 200 μm. (iii) 300 μm (iv) 100 μm. What is the flow in each channel? What is the effect of the location of each channel? What would the total flow rate be if the syringe pump is replaced with a pressure source with $P = 10^4$ Pa?.

2.1 Answers

This is mainly just a set of calculations to apply the formulas given in the slides. I am attaching an Excel sheet that does the actual calculations.

1. the highest Reynolds number is in the micro-channel: $Re = 0.3$. This is because the velocity increases in each tube as 1/area, while the Reynolds number scales with the length. That means that the fluid accelerates *more* than the length decreases, which implies that the Reynolds number increases as the size of the tube/channel decreases. Naturally, we are still in the low Reynolds number regime and the Stokes flows still apply

2. The pressure drops can simply be calculated from the two formulas $\mathcal{R}Q = \Delta P$. We find that the pressure drops in the syringe and in the tubes are negligible compared with the pressure drop in the micro-channel. The pressure drop in the microchannel is around $\Delta P = 10^4$ Pa, or 0.1 atm. So replacing the syringe pump with a pressure source applying 10^4 Pa would not change anything.

3. In the network, we have to use the electrical circuit analogs. The total resistance can be found by setting:

$$\frac{1}{R_{total}} = \frac{1}{R_i} + \frac{1}{R_{ii}} + \frac{1}{R_{iii}} + \frac{1}{R_{iv}} \simeq \frac{1}{3 \times 10^{13}} \quad (1)$$

   Note that R_{total} is smaller than all of the other resistances.

   The pressure across the system can therefore be obtained by setting $\Delta P = Q \cdot R_{total}$, which yields $\Delta P \simeq 10^3$ Pa. Then using again $Q_i = \Delta P / R_i$ for each path, we find the flow rate in each channel. Note that their sum will yield the total Q.

Now if we replace the syring pump with a pressure source pushing at $P = 10^4$ Pa again, we will multiply all of the flow rates by 10 since the resistance of each channel remains constant but the pressure drop across it increases ten-fold.

3 The effect of the presence of a bubble

Consider a microchannel such as the one shown in Fig. 2a. Imagine that an air bubble is stuck in the syringe. Its volume is 1 μL (ignore surface tension effects).

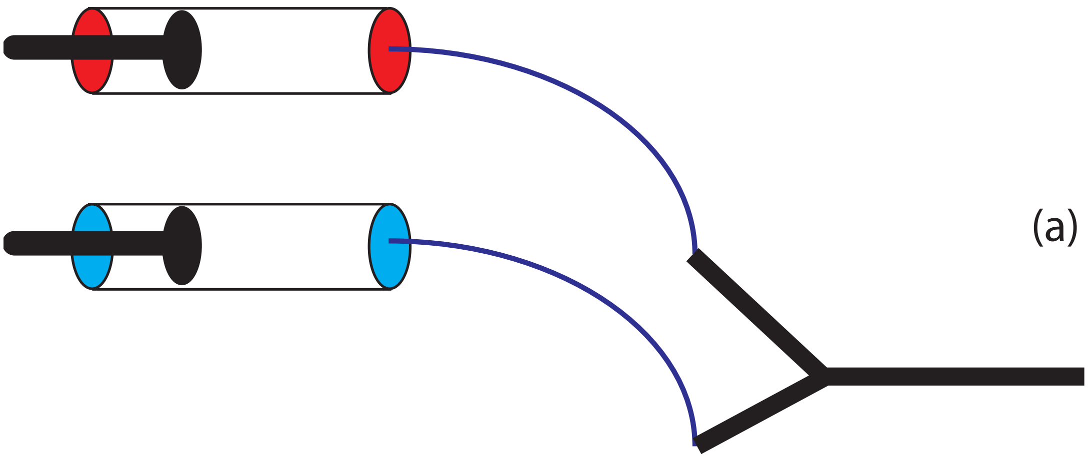
1. What is the radius that corresponds to this bubble volume?
2. By how much does the bubble compress for a flow $Q = 30$ nL/sec? How does that change if $Q = 300$ nL/sec? (Recall perfect gas law).
3. Now suppose that the flow is suddenly switched off and the bubble wants to return to the volume it has at atmospheric pressure. From the resistance of the channels calculated above and supposing that the pressure in the bubble is constant during this time (for example take the average value of pressure), estimate how long it takes for the flow to stop.
4. What would the "exponential" time be if we calculate it from the RC equivalent circuit?

`The moral of this story is that you should avoid having bubbles in your systems!`

3.1 Answers

1. The radius for 1 μL is around 1 mm
2. The ideal gas law states that $P_i V_i = P_f V_f$, with $P_i = 10^5$ Pa and $V_i = 10^{-9}$ m^3. So we get that $V_f = P_i V_i / P_f$, which yields $V_f(30) = 0.9$ μL and $V_f(300) = 0.5$ μL.
3. So we have to drain a volume equivalent to the change in bubble volume, at a pressure half the initial pressure, knowing that the resistance is 3×10^{14} Pa/m^3 as calculated above.
   We can calculate $Q = \Delta P/R$, which allows us to get the time required by calculating $(V_i - V_f)/Q$. This yields $\tau = 6$ s for the first case ($Q = 30$ nL/s) and $\tau = 3$ s for the second case.
4. In reality the process of draining is an exponential with characteristic time scale $\tau = RC$. C is given in the slides as $C_h = P_0 V_0 / P^2$, where P_0 is atmospheric pressure, $V_0 = 1$ μL is the volume at P_0, and P^2 is the pressure just when the syringe pump is turned off. This yields: $RC(30) = 3$ s and $RC(300) = 1$ s.

4 Mixing



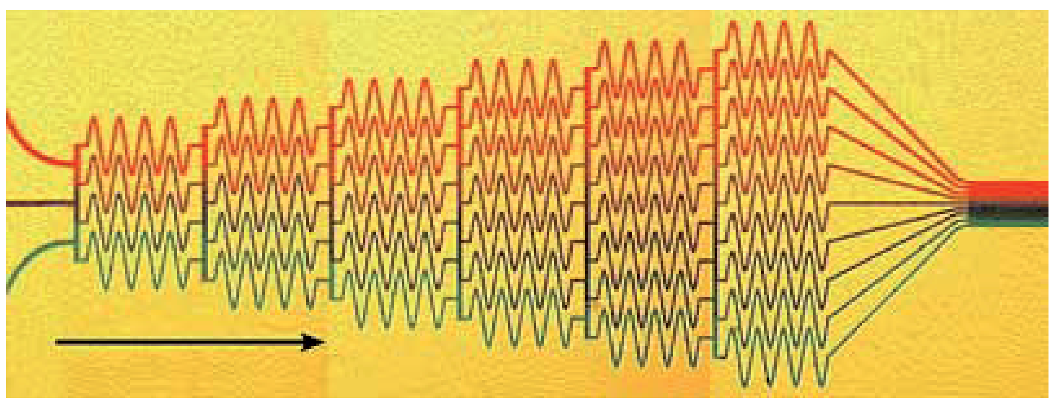


Fig. 3- Mixing in microchannels.

1. Consider the channel in Fig. 3a. The main channel has the same dimensions as the channel in Fig. 2a but it has two inputs. Eeach of the inputs is connected to a syringe pushing at $Q_1 = Q_2 = 15$ nL/sec, with one pushing blue dye and the other pushing red dye. Considering that the diffusion coefficients of the red and blue dyes are $D = 10^{-9}$ m^2/s, what is the percentage of the two two dyes that mixes in a channel that is 2 cm long?

2. The channel that is shown in part (b) was designed to generate a gradient of dye in the downstream channel (see the wide channel on the right hand side). How does it work?

4.1 Answers

1. The distance that is diffused by each species into the other one can be estimated by saying that $w_d = \sqrt{D\tau}$, where $D = 10^{-9}$ m^2/s and τ is ths time to travel down the channel. $\tau = L/U = L/(Q/wh) = 1.33$ s. The distance that each of the species diffuses is $w_d = 36$ μm. So we can expect that 36% of the liquid has been mixed.

2. The channel is supposed to make a gradient of color. The three colors initially enter the channel and mix two-by-two at the first junction. Each zig-zag allows the colors to mix thoroughly, before reaching the next junction where the neighboring colors mix two-by-two again. Note that each exit of a channel leads to a bifurcation: Half of the liquid goes left and half goes right. This continues until all of the liquid reaches the wide channel, in which we now have a continuous spectrum of colors.