Micro-scale flo Charles N. Baroud

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What's so special about micro-flows?

Macro-scale flows

Macro-scale flows

Vortices

Micro-scale flows

Macro-scale flows Time varying

Micro-scale flows

Regular!

Flow lines: DNA chamber **Protein chamber**

Control lines: **Neck valve Sandwich valve Button valve**

Macro-scale flows Turbulent

Macro-scale flows Regular!

What happens at micro-scale?

- No new physics (sorry!)
- Relative strength of different forces is changed:
- e.g. viscosity becomes dominant over inertia
- Surface to volume ratio grows
- surface effects become dominant over volumetric effects

Surface vs. volume effects

Surface vs. volume effects

- Weight $\sim L^3$
- Capillary force *F* : γ
	- Surface tension *x* (*L*)
- When *L* decreases:

weight << F_y

Outline

- Viscous flows The Reynolds number
- Properties of the Stokes Equation
- Hydraulic circuit analysis
- Molecular diffusion in micro-flows

The fluid particle

Small compared with system size Large compared with molecular scales

How long can we still talk about a fluid?

The fluid particle *•* Viscosity becomes dominant compared with inertial effects *•* The surface to volume ratio increases, meaning that surface effects dominate volume

Can change velocity in two ways:

- ^o By changing in time
	- By moving in space

 $\frac{D \vec{u}}{D t}$ = $\partial \vec{u}$ $\frac{\partial u}{\partial t} + \vec{u} \cdot \nabla \vec{u}$ \vec{u} Define the «material» derivative

The momentum equation eq **ation**

Navier-Stokes equation for space (*L*₎, and taking the material properties of the fluid \mathbf{p} and \mathbf{p} and \mathbf{p} and \mathbf{p} and \mathbf{p} This leads to the definition of **Research**

The momentum equation eq **ation**

Navier-Stokes equation

$$
\rho \left[\begin{array}{cc} \partial \vec{u} \\ \overline{\partial t} \end{array} + \ \vec{u} \cdot \vec{\nabla} \vec{u} \ \right] = - \vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}
$$

volume of the fluid, not on the boundaries. Many possible solutions!

The momentum equation eq The momentum equation *Dt* [∂]*^t* ⁺ ! A force balance on this particle leads to the Navier-Stokes Equation:

Navier-Stokes equation a formulation ! ²!

ρ*UL*

$$
\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}
$$

Define «characteristic» scales $\bigcap_{\alpha} f_{\text{in}} \circ \theta$

The Reynolds number. For space (*L*), and the material properties $\mathbf{P}_{\mathbf{A}}$ This leads to the definition of *Re*: The Reynolds number:

The momentum equation *Re* = *µ*

Dimensionless Navier-Stokes equation Dimensionless Navier-

$$
Re\left[\left.\frac{\partial\vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right]\right] = -\vec{\nabla}p + \vec{\nabla}^2\vec{u}
$$

Here: Length, velocity, and physical
hidden in the Reynolds number. *u* pur \mathfrak{m} *divelopment* and other Here: Length, velocity, and physical parameters have all been

The behavior b \overline{a} The behavior becomes a function of a single parameter: Re *Re*

Micro-scale flow

µm diameter at 100 µm/s Water flowing around a cylinder of 100 100

$$
\mu = 10^{-3} \text{ Pa s} \n\rho = 10^{3} \text{ kg/m}^{3} \nU = 100 \text{ }\mu\text{m/s} \nL = 100 \text{ }\mu\text{m}
$$
\n
$$
R = \frac{\rho UL}{\mu} = 10^{-2}
$$

$$
\rho = 10^3 \text{ kg/m}^3
$$

\n
$$
Re = \frac{\rho UL}{\mu} = 10^{-2}
$$

\n
$$
J = 100 \text{ }\mu\text{m/s}
$$

understand µ-scale flows. **Re** ± **b**e a **P** solutions of the N-S **Equation at low Re Will allow** Understanding the solutions of the N-S equation at low *Re* will allow us to

Low Reynolds number *Re* = **r** *r*

$$
Re\left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u}\right] = -\vec{\nabla}p + \vec{\nabla}^2 \vec{u}
$$
\nReplace a nonlinear
\nequation with a linear one

y
Pressent Replace a equation with a linear one **Equation with a linear one**

$$
\boxed{\vec{\nabla} p = \nabla^2 \vec{u}}
$$

The Stokes Equation

Characteristics of Stokes Flows

• Laminar: no turbulence and no advective mixing

Characteristics of Stokes Flows

- Laminar: no turbulence and no advective mixing
- Linearity implies:
	- ‣ Unique solution for given boundary conditions
	- ‣ Reversibility: A change of *p* → -*p* switches *u* → *-u*
	- ‣ Streamlines are not modified if total flow rate changes
	- ‣ Superposition of solutions: When boundary conditions are added, solution is sum of individual solutions
- Stokes flows correspond to a minimum of energy dissipation

Reversibility implies symmetry

Reversibility implies symmetry

Reversibility implies symmetry **u** (1) \mathbf{u} (1) \mathbf{u} (1)

$\overline{1}$ This flow is not the same as its reverse

Reversibility implies symmetry **u** (1) \mathbf{u} (1) \mathbf{u} (1)

$\overline{1}$ This flow is not the same as its reverse

1 This flo This flow cannot be solution to Stokes equation

Increasing complexity of flows with increasing Reynolds number: flow past a circular cylinder

Classical solutions of Stokes Flows

Systematically change the aspect ratio of the cavity: note appearance of one or more eddies

G.I. Taylor demonstrates reversibility

Interpretation of the Reynolds number I I_n

- *•* Inertial (stagnation) pressure stress on an object: *p* ∼ ρU^2 • Stagnation pressure on a solid ρ $\sim \rho U^2$
- Shear stress due to the velocity gradient

Interpretation of the Reynolds number II *D*! *u Dt* $\overline{}$ A force balance on this particle leads to the Navier-Stokes Equation:

• Advection time *•* Advection time: Δt

$$
\tau_{adv}\sim \frac{L}{U}
$$

Interpretation of the Reynolds number II tation of the Reyn

ται γ
γ αι .
t **Ly transmitted** τ*dif f* ∼ Change in boundary velocity transmitted *via* a diffusive process

• Viscous diffusion time \bullet Viscous diffusion time

$$
\tau_{diff} \sim \frac{L^2 \rho}{\mu} = \frac{L^2}{\nu}
$$

$$
\frac{L^2}{\nu} \qquad \qquad [\nu] = \frac{L}{T^2}
$$

Kinematic viscosity

3 Two basic flows basic flows
3 Two basic flows basic flows basic flows
3 Two basic flows Boundary moving below a highviscosity oil

Interpretation of the Reynolds number II *^U* (15) *•* Advection time:

· Viscous diffusion time si *µU/L* • Viscous diffusion time

T_{diff} ~
$$
\frac{L^2 \rho}{\mu} = \frac{L^2}{\nu}
$$
 $[\nu] = \frac{L}{T^2}$

• Advection time *•* Advection time: *•* The ratio yields:

$$
\tau_{adv} \sim \frac{L}{U}
$$

3.1 Couette flow

$$
\boxed{\frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L} \sim \frac{UL}{\nu} = Re}
$$

Back to N-S. to Stokes equation Back to N-S. to Stokes equation: **Back to**

$$
\rho \left[\begin{array}{cc} \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \end{array} \right] = -\vec{\nabla}p + \mu \vec{\nabla}^2 \vec{u}
$$
\n
$$
\left[\begin{array}{c} \n\text{Negligible if } \frac{p}{\sigma} \sim \frac{\rho U^2}{\mu U/L} << 1 \\
\text{(Viscous dominate over inertial stresses)} \\
\text{Negligible if } \frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L} << 1 \\
\text{(fluid adjusts immediately to variations in BC)}\n\end{array} \right]
$$

Two basic flows

- Boundary driven flow in a gap (Couette flow)
- Pressure driven flow in a tube (Poiseuille flow)

Couette flow Couette flow

$$
\mu \frac{\partial^2 u}{\partial y^2} = 0
$$

$$
u(y=0) = 0 \quad u(y=h) = u_H
$$

$$
u=u_H\left(\frac{y}{h}\right)
$$

 $\int x y \, dy \, dy$ that is applied by the bottom wall is applied by the bottom wall is $\int x^2 y dx$ $\boldsymbol{F} = A \boldsymbol{\cdot} \tau_{xy}.$ \mathbf{r} that is applied by the bottom wall is applied by the bottom wall is applied by the bottom wall is: $\tau_{xy} = \mu$ ∂u ∂y $=$ μ u_H $\frac{F}{h}$ $F = A$. Shear stress on the wall: an the wall.

u = *u^H* late *v*⁽ *a p*^{*x*}(*x*) *a p*^{*x*}(*x* ∂*u* = *µ* For a plate of area A *^h* (22) \mathbf{r}

$$
F = A \cdot \tau_{xy}
$$

Force on a falling sphere ∂*u u^H* Force on a falling sphere in Store in Stocker in Stocker where \mathbf{e} and since due to the shear stress on its surface. If the sphere is settling by its settling by its surface. If the sphere is set F_{A} sphere in Stockes flow: A sphere that is settled in a viscous flow: A sphere that is settled in a viscous flow: A sphere that is settled in a viscous flow: A sphere that is set of that is set of the sphere would feel a drag force due to the shear stress on its surface. If the sphere is settling by its surface. If the sphere is set

$\frac{1}{d}$ $\frac{d - \sigma \theta \mu \mu \nu}{d}$ Settling velocity $U =$ 2 9 *g*∆ρ *µ* R^2

Verify drag force scaling

Left sphere twice as big as right sphere

Poiseuille flow Doiseuille flow 3.2 Poiseuille flow Now consider the flow in a circular pipe, driven by a circular pipe, driven by a pressure difference between t
The upstream of the upstream of upstream of upstream the upstream of the upstream of the upstream of the upstr \mathbf{P}

and downstream ends. We will assume that *p*(*z*) varies linearly so that Assume constant pressure gradient and downstream ends. We will assume that *p*(*z*) varies linearly so that and downstream ends. We will assume that Δ *z* α in the *p*(*z*) varies linearly so that *p*(*z*) varies linearly so

$$
\nabla p = \frac{\partial p}{\partial z} = cst
$$

Velocity inside cylindrical tube of radius *R*

$$
u_z = \frac{-1}{4\mu} \frac{\partial p}{\partial z} (R^2 - r^2)
$$

Flow rate:
$$
Q = -\frac{\pi R^4}{8\mu} \frac{\partial p}{\partial z}
$$

Tube of length *l* T is a Newtonian fluid obeys the velocity field T Tube of length l

$$
Q = \frac{\pi R^4}{8\mu L} \Delta p = \frac{\Delta p}{\mathcal{R}}
$$

$$
\mathcal{R} = \frac{8\mu L}{\pi R^4}
$$
 the fluids equations t
know the flow rate

Face tension You don't need to solve $\overline{\text{SUT}}$ ents give Marangoni stresses #*^m* **KIIOW LIIE** Hydrodynamic resistance You don't need to solve $\frac{1}{2}$ and the salve $\frac{1}{\pi R^4}$ (310) when the flow rate the fluids equations to

Microchannel **Microchannel** navier-Stokes en 1990, which essentially representations, which essentially representations, which essentially In the case of a rectangular channel, the Poiseuille solution is not exact solution is not exact. The exact solution is not exact sol

The analogy with electrical circuits goes beyond simple resistance. We can make the

$$
Q \simeq \frac{wh^3}{12\mu L} \left[1 - 6\left(\frac{2}{\pi}\right)^5 \frac{h}{w}\right] \Delta p
$$

Hydrodynamic resistance If
$$
h < w
$$

$$
\mathcal{R} = \frac{12\mu L}{wh^3} \left[1 - 6\left(\frac{2}{\pi}\right)^5 \frac{h}{w} \right]^{-1} \qquad \mathcal{R} = \frac{12\mu L}{wh^3}
$$

following analogies: Strong dependence on *h*

Analogy with electrical circuits Naviers-Stocker extra ex Analogy with electrical

Compliance with electrical circuits goes beyond simple resistance.

Compressibility is a way to store pressure: Fluidic capacitance

The Compliance associated with an a Compliance associated with an air bubble:

$$
C_h = \frac{p_0 V_0}{p^2} \longrightarrow \frac{\partial P}{\partial t} = \frac{1}{C_h} Q
$$

R-C circuit

Hydrodynamic resistance

Compliance

∂*P* ∂*t* $=$ *C^h* $-Q$ ∂P ∂*t* = 1 C_h *Q* (36) ∂*P* U^t \overline{a} $\overline{1}$ *C^h* α

Therefore: \sum harafora:

$$
\frac{\partial P}{\partial t} = \frac{1}{\mathcal{R}C_h}P
$$

$$
P(t) \sim e^{-t/RC_h}
$$

P(*t*) = *e*−*t/RC^h* (38) *P*(*t*) = *e*−*t/RC^h* (38) that the system will respond very slowly Large resistance and large compliance imply

A few words about diffusion

A few words about diffusion ∂*t* ff *RC^h P* (37)

Molecular scale model:

A molecule performs a random walk with a certain step size during every time step.

during every time step.

Mean field model: The "mean field" description is the "mean field" description is the "mean field" of diffusion is the "mean field" of diffusion is the "mean" of diffusion is the "mean" of diffusion is the "mean" of diffus

A chemical species is transported «down» the concentration gradient A chemical species is transported «down» the cone

$$
\frac{\partial C(x,t)}{\partial t} = D \frac{\partial^2 C(x,t)}{\partial x^2}
$$

Useful solutions ∂*C*(*x, t*) ⁼ *^D* [∂]2*C*(*x, t*)

Diffusion quickly evens out short-wave variations ∂*t* [∂]*x*² (39) ut short-wave variatic

Diffusion in a T-channel $T_{\rm eff.}$ M. Squires and S. R. \sim R

$\text{Im} \Omega = \text{Im}$ How far will the species diffuse?

 $\mathbb{E} \left[\left\| \mathbf{f} \right\| \right]$ Time to diffuse whole width $\tau_{diff} \sim$ w^2 $\frac{a}{D}$ $\tau_{diff} \sim$ w^2 $\frac{w}{D}$

 Z^{zone} istance travelled during this time $Z \sim U_0^{\gamma}$ diff $\sim U_0^{-1}$ $\frac{v^2}{\sqrt{v^2}}$ *^D* (42) $Z \sim U_0 \tau_{diff} \sim U_0$ w^2 $\frac{a}{D}$

How many channel widths?

How many
channel widths?
$$
\boxed{\frac{Z}{w} \sim \frac{U_0 w}{D} = Pe}
$$
 Peclet number as ratio
of two lengths

How many $Z = U_0w = P_e$ Peclet number as ratio channel widths: $\begin{bmatrix} w & D & \end{bmatrix}$ of two lengths \mathbf{F}_{max} exploits the different rates at \mathbf{v} = *P e* (43) of two lengths

Pe vs. Re **Property** \mathbb{R}^p

$$
\boxed{\frac{Z}{w} \sim \frac{U_0 w}{D} = Pe}
$$

$$
\left[\frac{Z}{w} \sim \frac{U_0 w}{D} = Pe\right] \qquad \qquad \frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L} \sim \frac{UL}{\nu} = Re
$$

diffusive transport. It has a very similar structure to the Reynolds number, but where α Reynolds number as a Peclet number as a Peclet number as a Peclet number of as Peclet number for momentum transfer