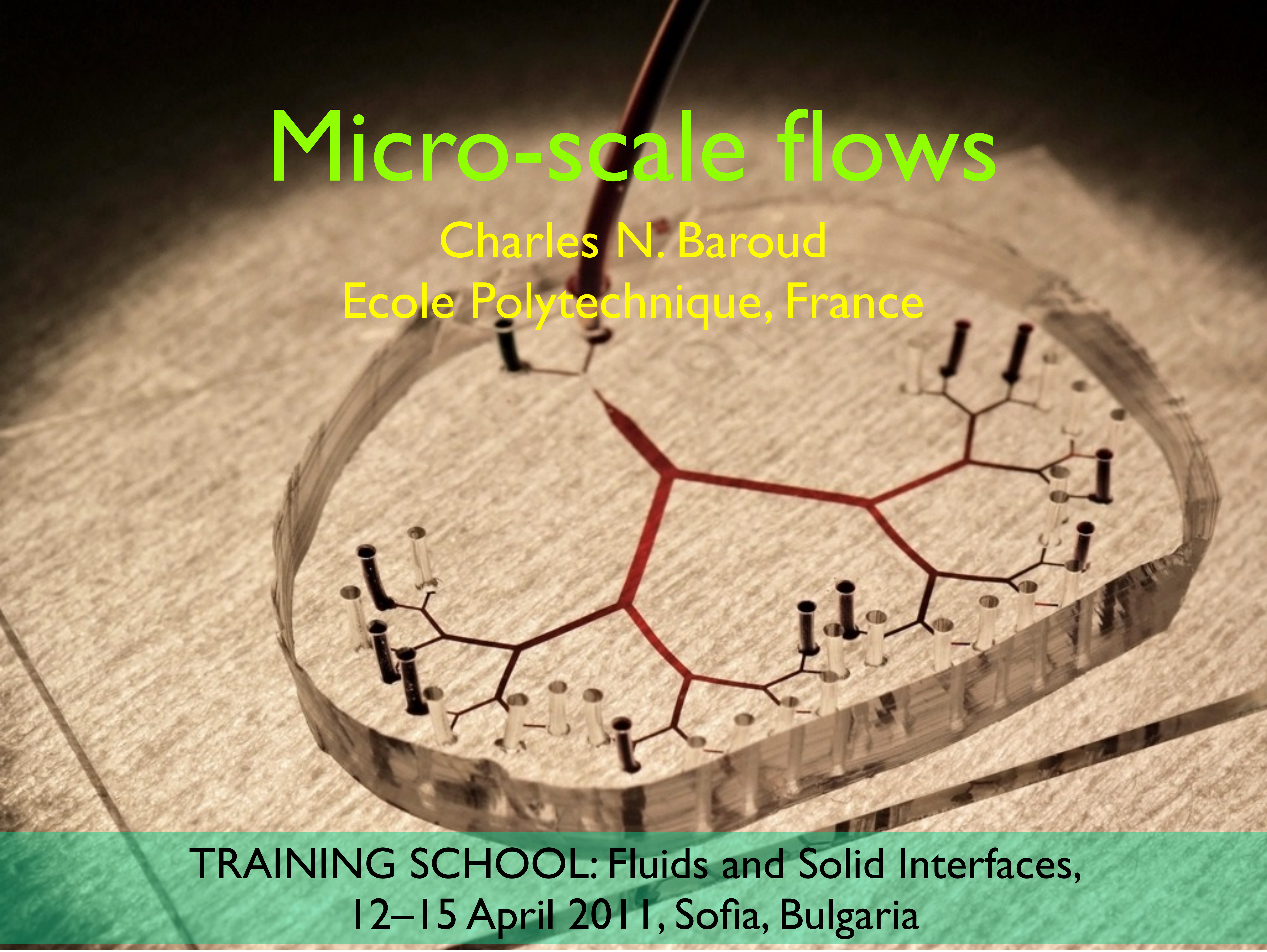

Micro-scale flows

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Ecole Polytechnique, France

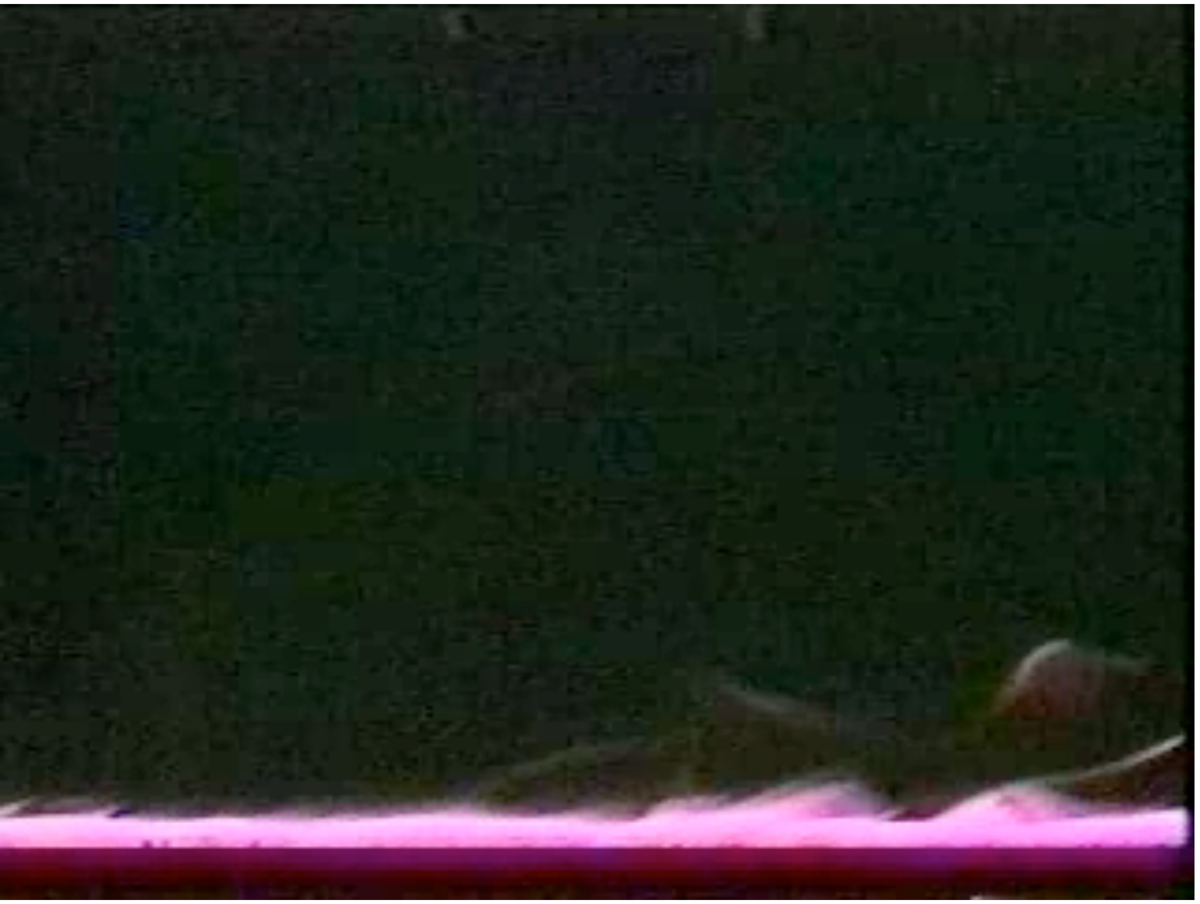
TRAINING SCHOOL: Fluids and Solid Interfaces,
12–15 April 2011, Sofia, Bulgaria

What’s so special about micro-flows?

Macro-scale flows

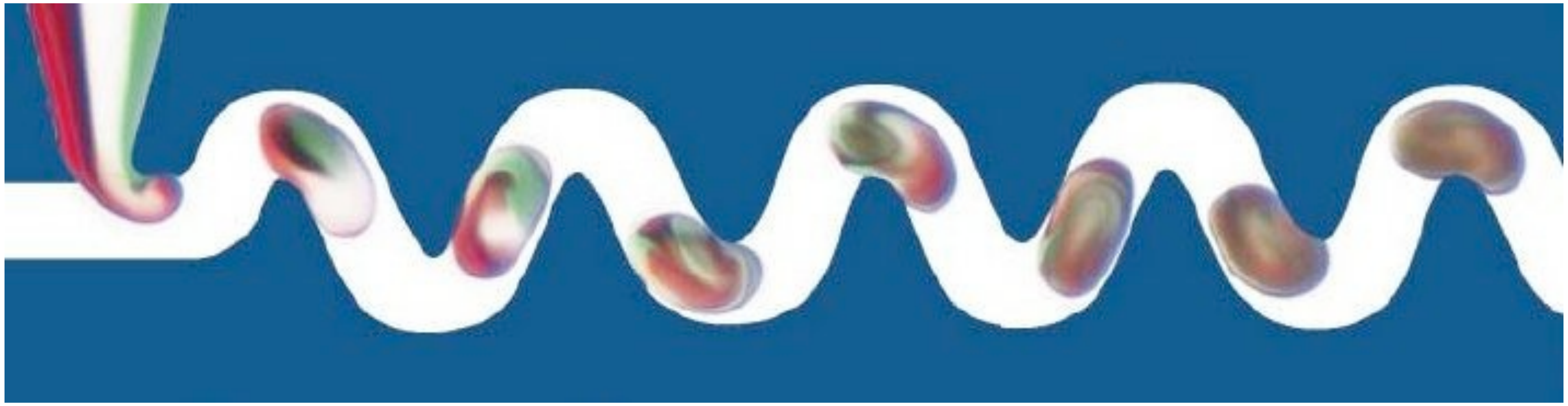
Micro-scale flows

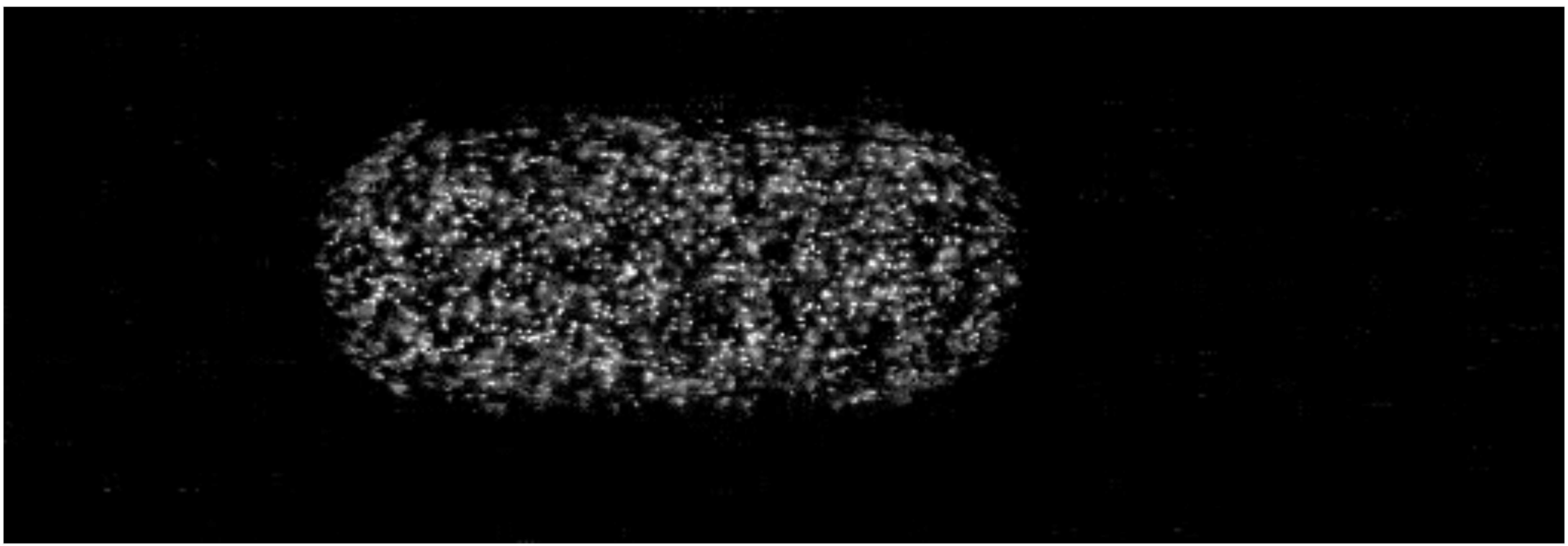
Macro-scale flows



Vortices

Micro-scale flows



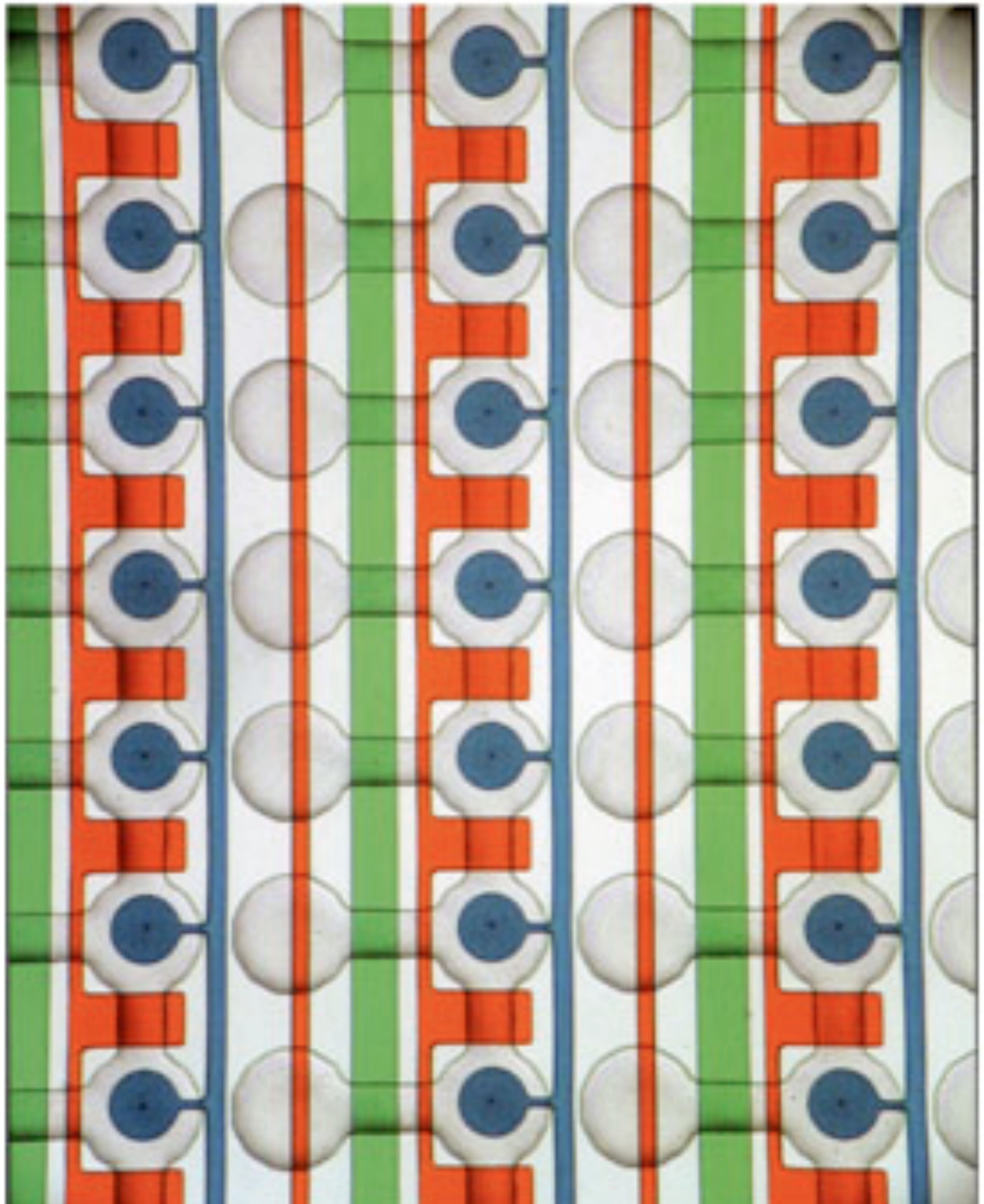


Macro-scale flows

Time varying

Micro-scale flows

Regular!



Flow lines:

DNA chamber

Protein chamber

Control lines:

Neck valve

Sandwich valve

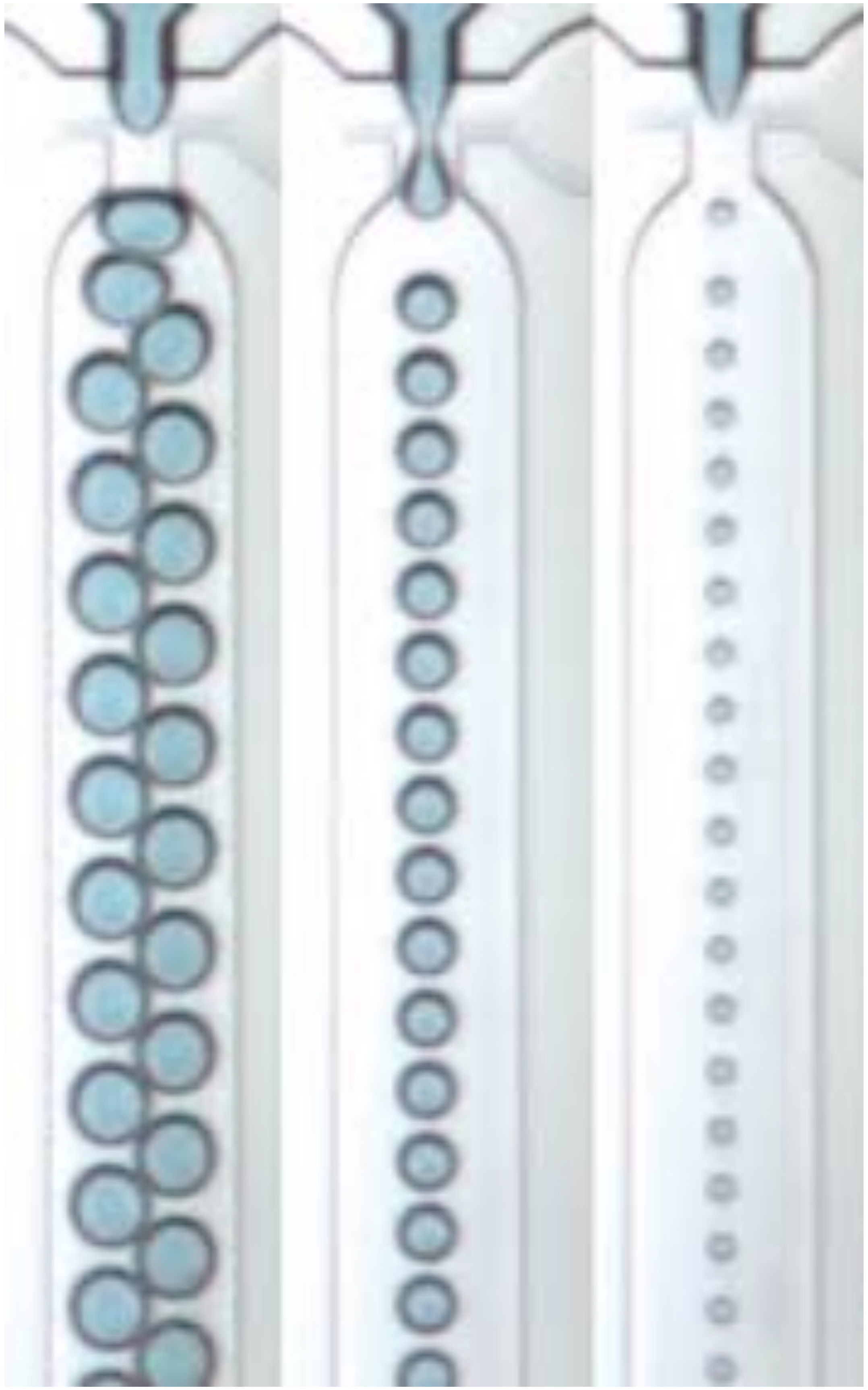
Button valve

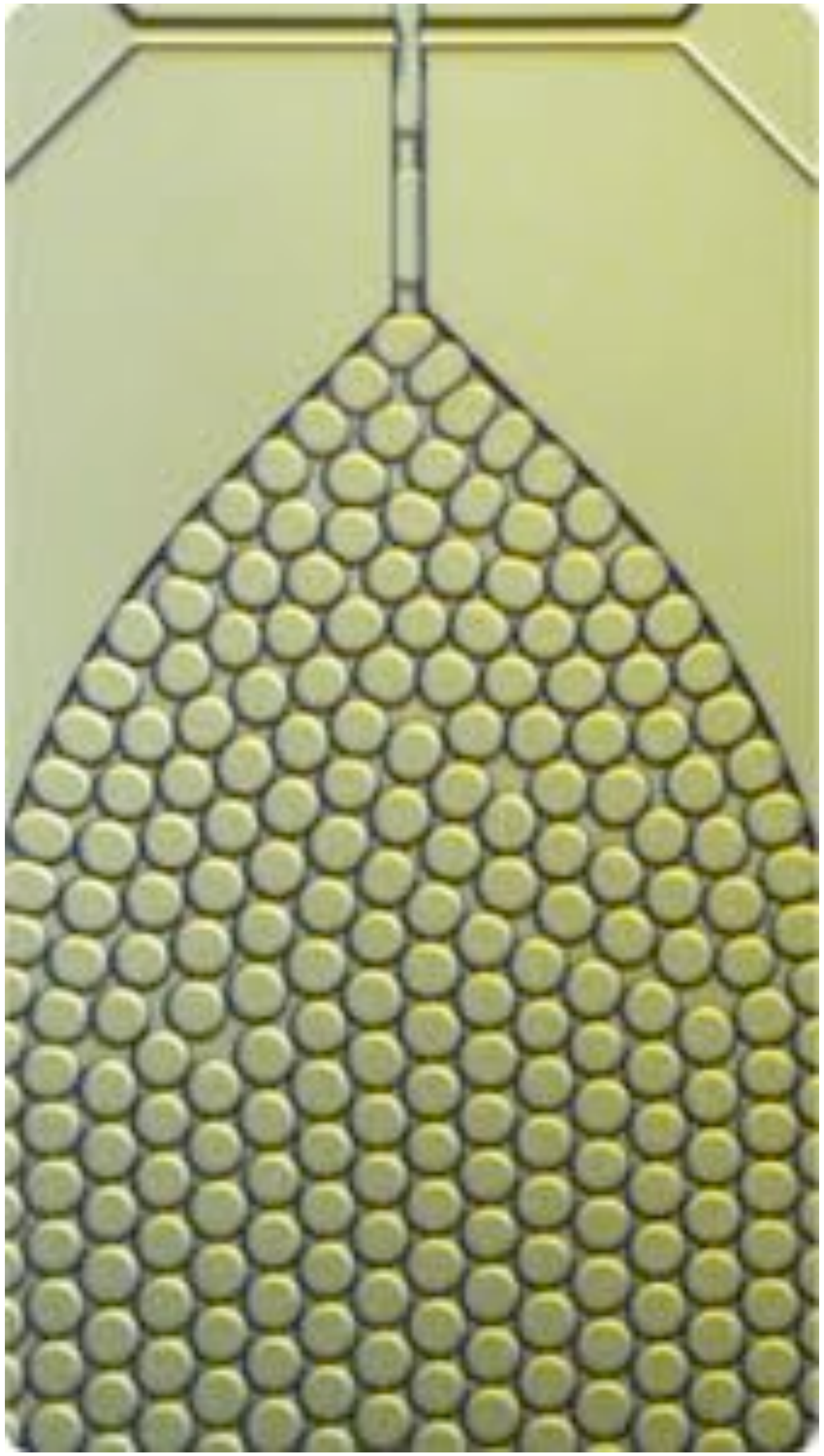
Macro-scale flows

Turbulent

Macro-scale flows

Regular!





What happens at micro-scale?

- No new physics (sorry!)
- Relative strength of different forces is changed:

➡ e.g. viscosity becomes dominant over inertia

- Surface to volume ratio grows

➡ surface effects become dominant over volumetric effects

Surface vs. volume effects

Surface vs. volume effects

- Weight ~ L^3
- Capillary force F_γ:

  Surface tension *x* (*L*)
- When *L* decreases:

  weight << F_γ

Outline

- Viscous flows - The Reynolds number
- Properties of the Stokes Equation
- Hydraulic circuit analysis
- Molecular diffusion in micro-flows

The fluid particle

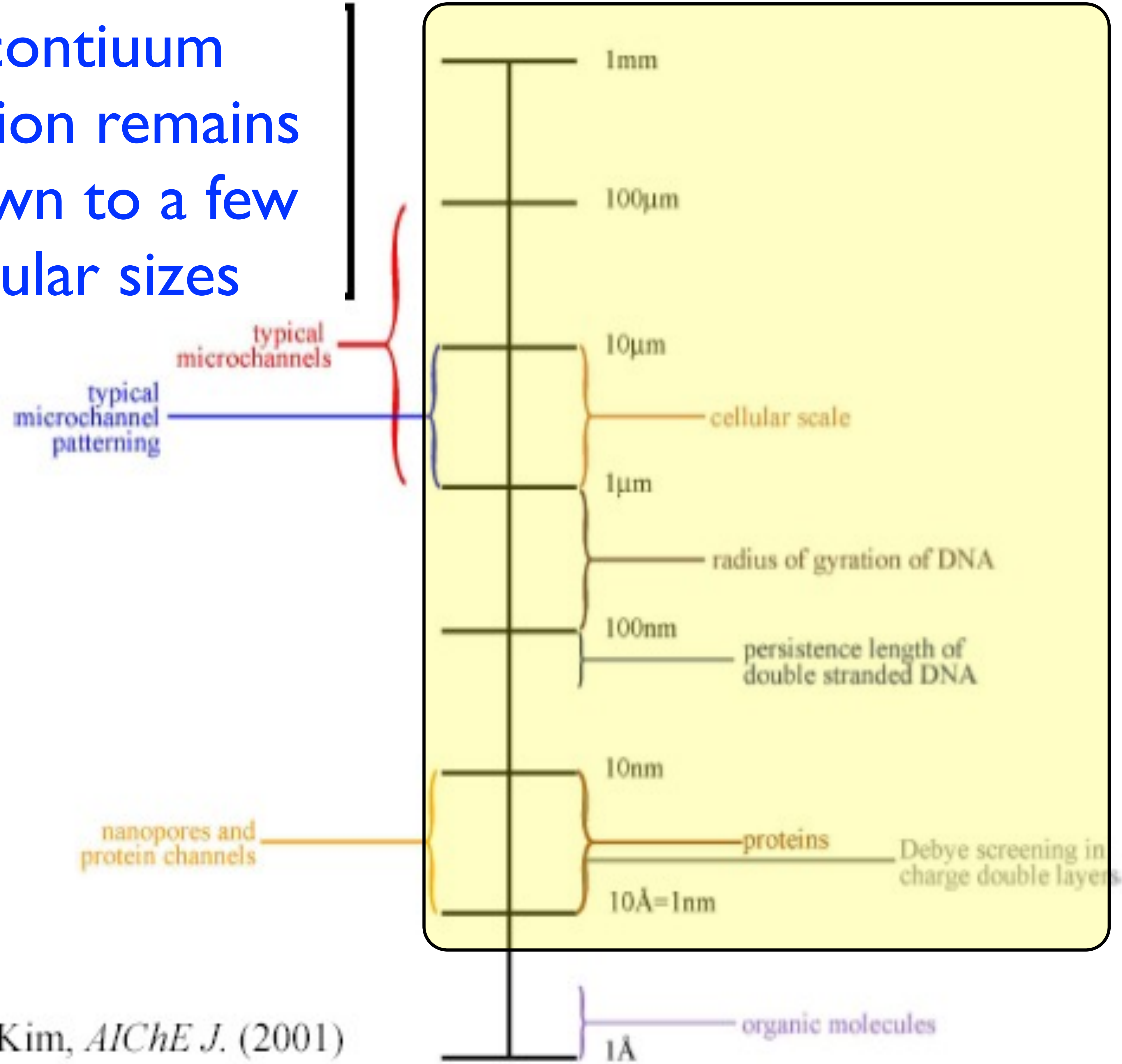
Small compared with system size

Large compared with molecular scales

How long can we still talk about a fluid?

The contiuum assumption

The contiuum assumption remains valid down to a few molecular sizes



Ref: Stone and Kim, *AIChE J.* (2001)

The fluid particle

Can change velocity in two ways:

- By changing in time
- By moving in space

Define the «material» derivative

$$\frac{D\vec{u}}{Dt} = \frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \nabla \vec{u}$$

The momentum equation

«Newton's law» for a fluid particle

simplify

$$\rho\left[\frac{\partial\vec{u}}{\partial t}+\vec{u}\cdot\vec{\nabla}\vec{u}\right]=-\vec{\nabla}p+\mu\vec{\nabla}^2\vec{u}+\cancel{\vec{F}}$$

Pressure gradient

Viscous stress

other body forces

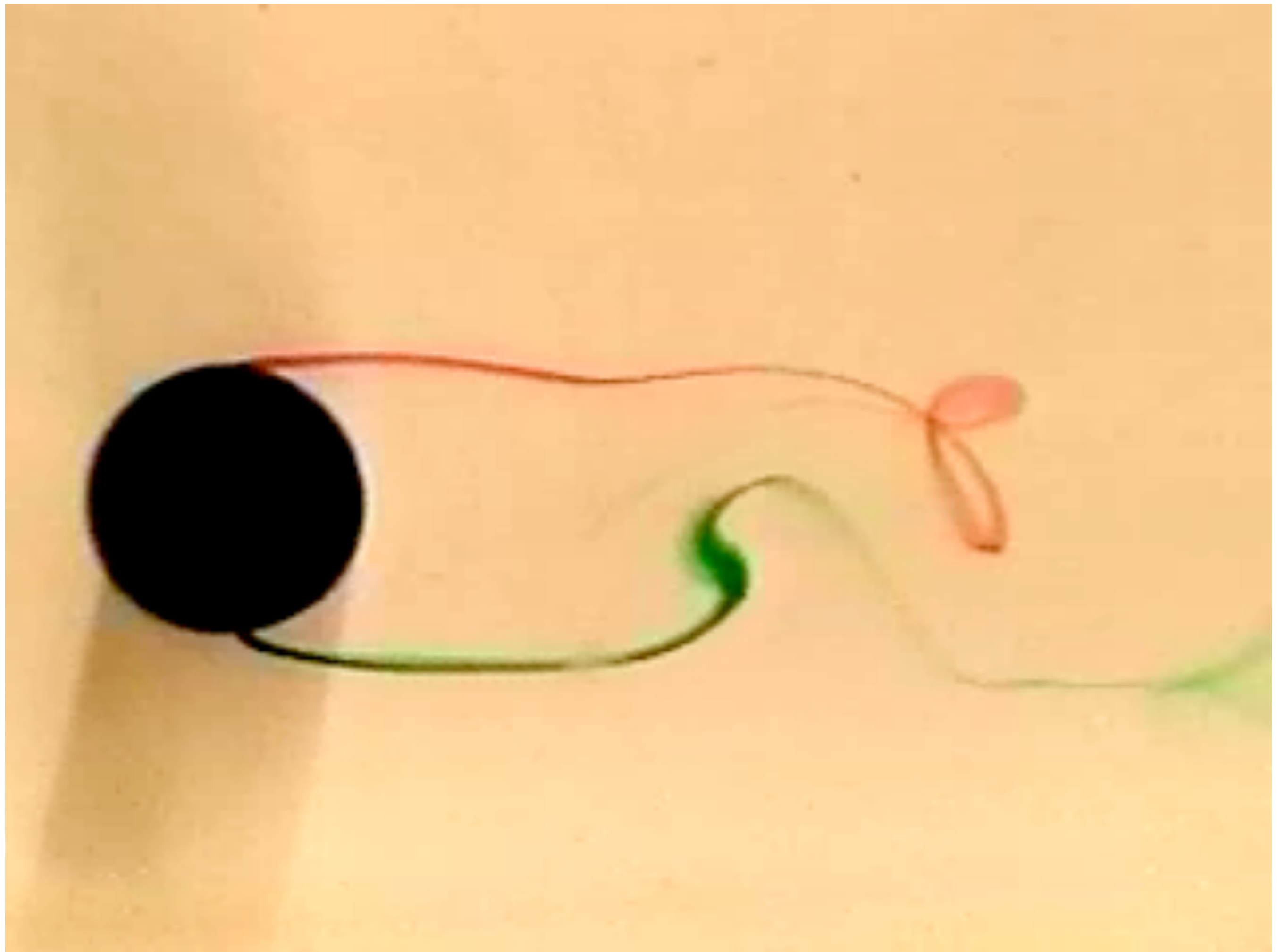
Navier-Stokes equation

The momentum equation

Navier-Stokes equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$$

Strongly nonlinear



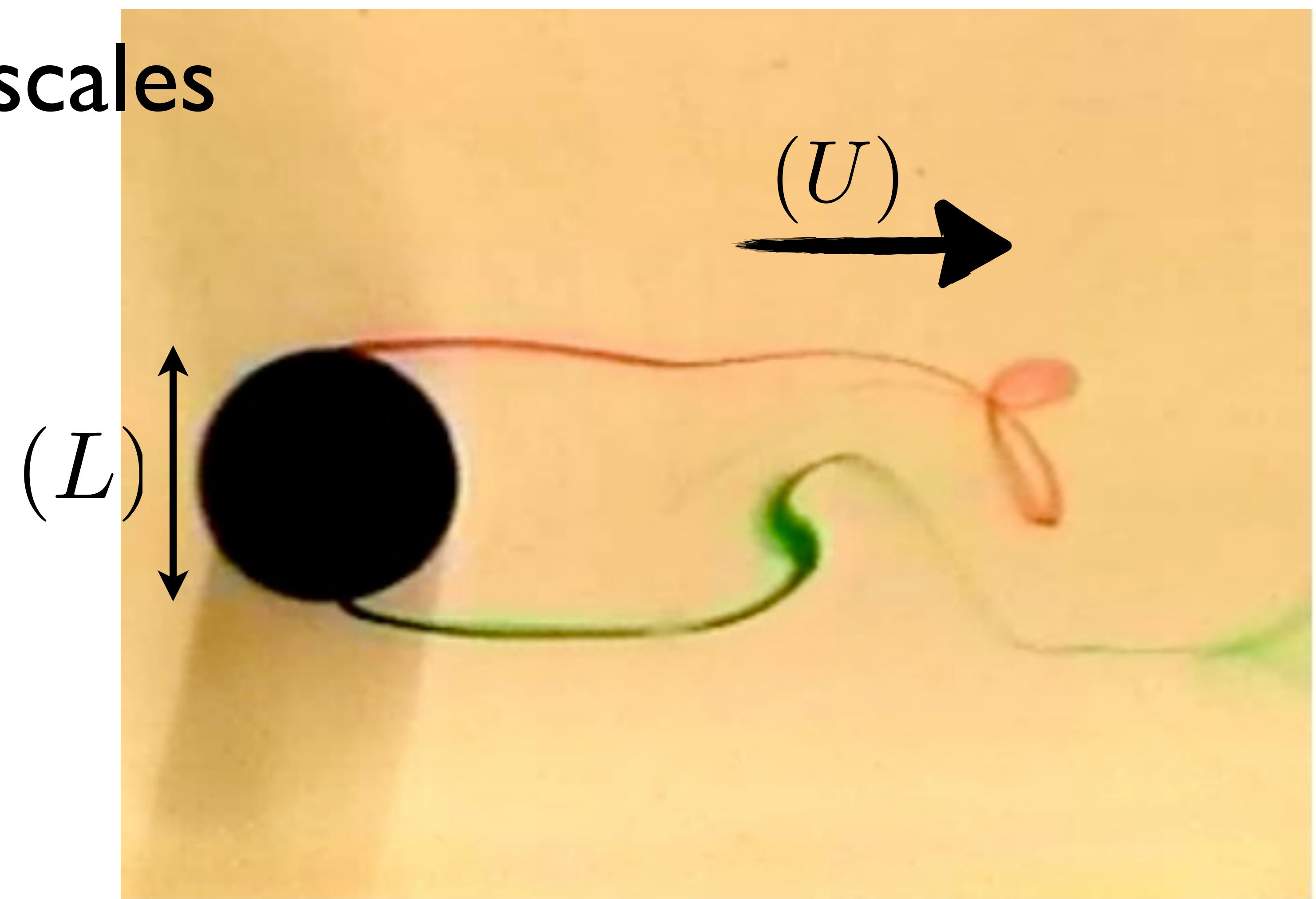
Many possible solutions!

The momentum equation

Navier-Stokes equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$$

Define «characteristic» scales



The Reynolds number:

$$\boxed{Re = \frac{\rho U L}{\mu}}$$

Dimensionless

The momentum equation

Dimensionless Navier-Stokes equation

$$Re\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right] = -\vec{\nabla}p + \vec{\nabla}^2\vec{u}$$

Here: Length, velocity, and physical parameters have all been hidden in the Reynolds number.

The behavior becomes a function of a single parameter:

$$Re$$

Micro-scale flow

Water flowing around a cylinder of 100 μm diameter at 100 μm/s

$$
\begin{aligned}
\mu &= 10^{-3}\ \mathrm{Pa\ s} \\
\rho &= 10^{3}\ \mathrm{kg/m^3} \\
U &= 100\ \mu\mathrm{m/s} \\
L &= 100\ \mu\mathrm{m}
\end{aligned}
$$

$$
Re = \frac{\rho U L}{\mu} = 10^{-2}
$$

Understanding the solutions of the N-S equation at low *Re* will allow us to understand μ-scale flows.

Low Reynolds number

$$Re\left[\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\vec{\nabla}\vec{u}\right] = -\vec{\nabla}p + \vec{\nabla}^2\vec{u}$$

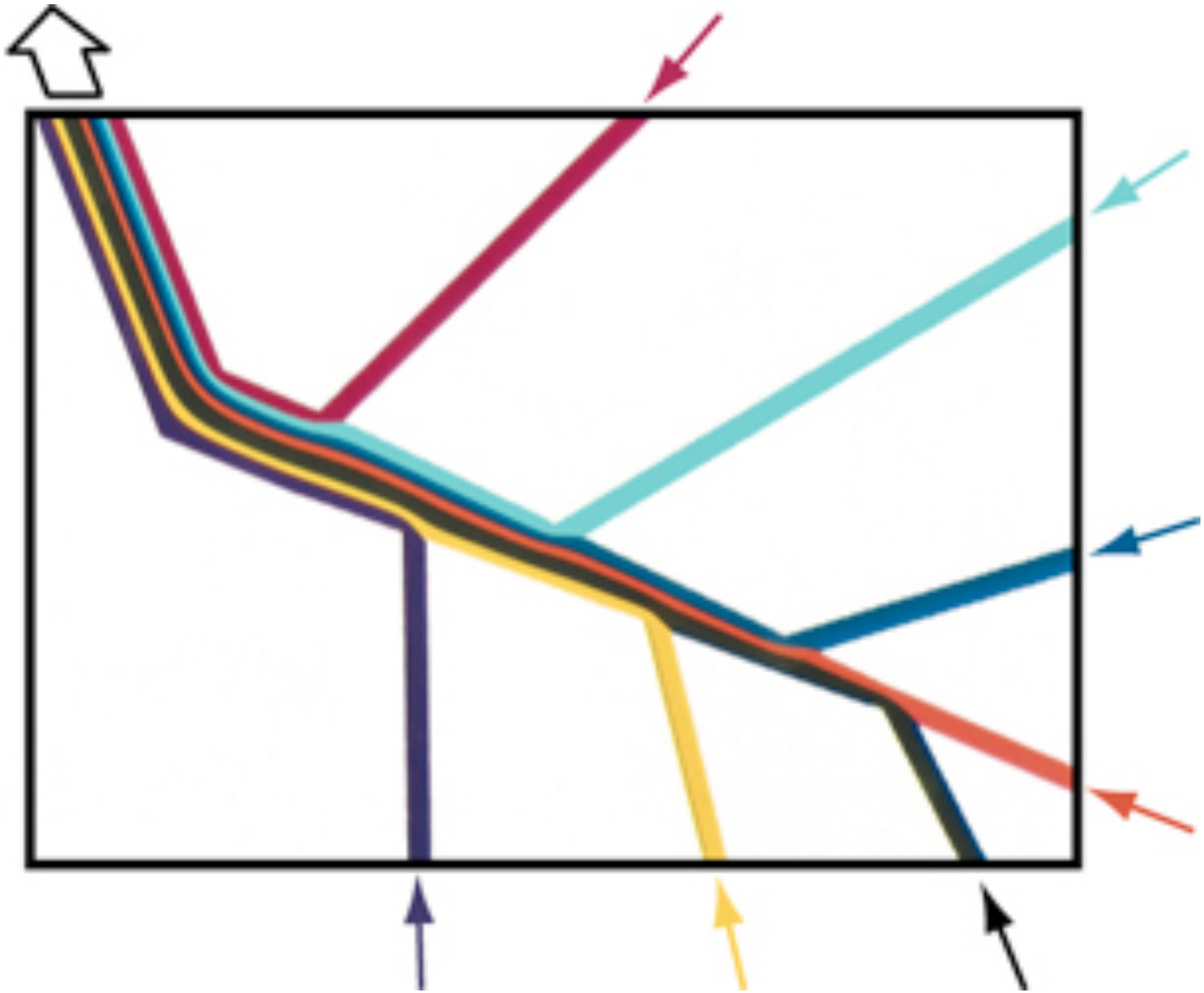
Replace a nonlinear equation with a linear one

$$\boxed{\vec{\nabla}p = \nabla^2\vec{u}}$$

The Stokes Equation

Characteristics of Stokes Flows

- Laminar: no turbulence and no advective mixing

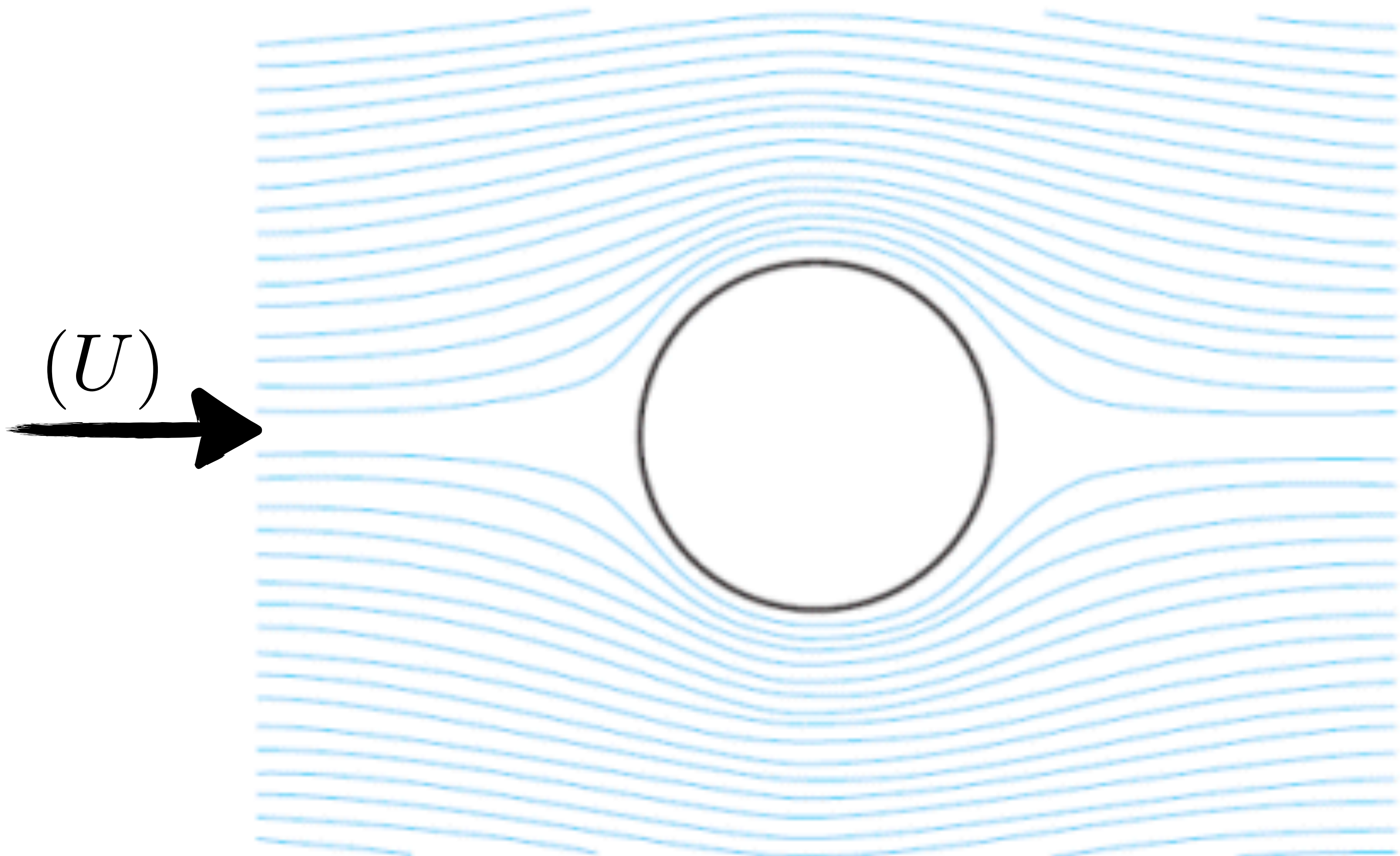


Whitesides lab, Harvard, 1998

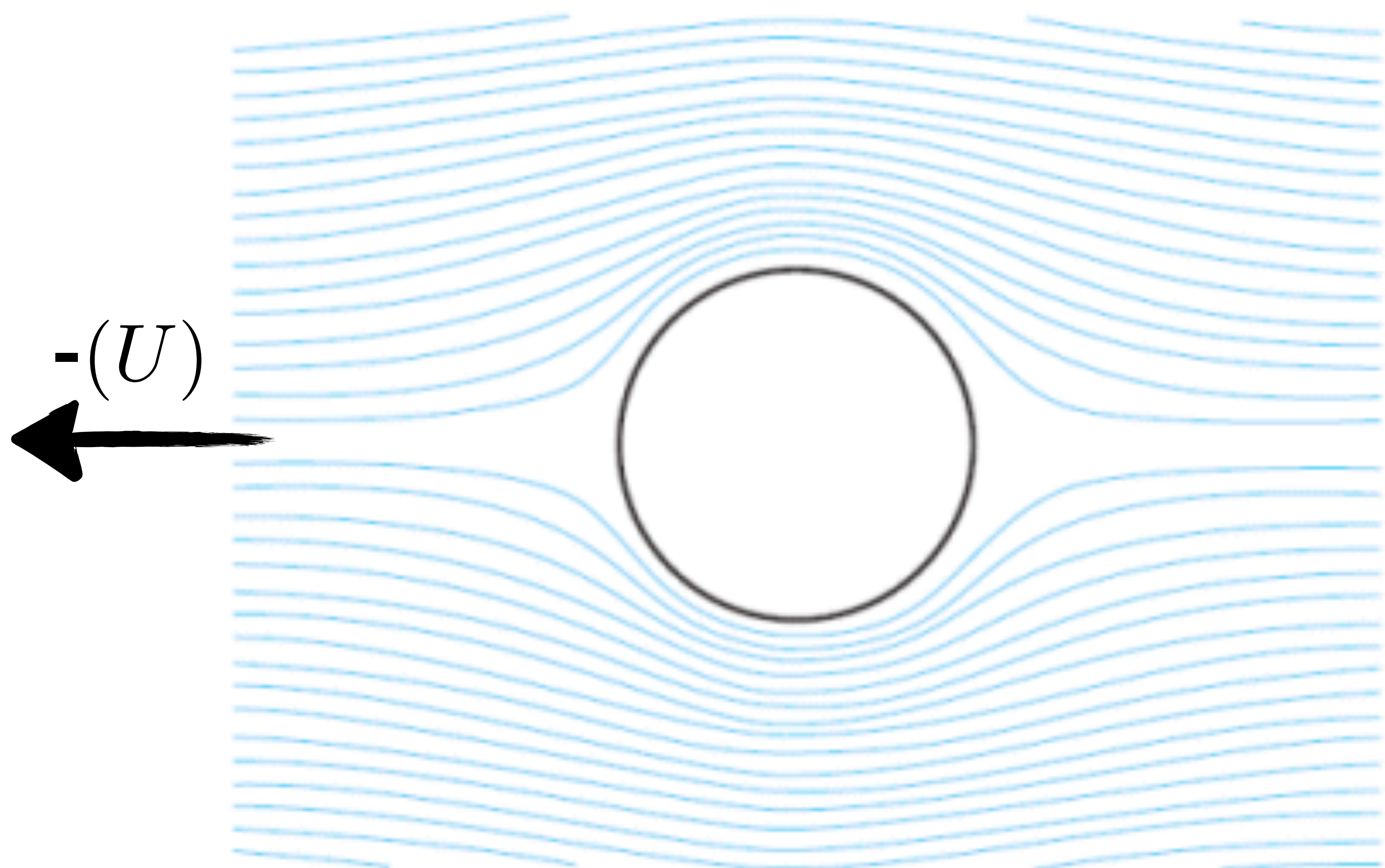
Characteristics of Stokes Flows

- Laminar: no turbulence and no advective mixing
- Linearity implies:
  - Unique solution for given boundary conditions
  - Reversibility:
    A change of $p \rightarrow -p$ switches $u \rightarrow -u$
  - Streamlines are not modified if total flow rate changes
  - Superposition of solutions:
    When boundary conditions are added, solution is sum of individual solutions
- Stokes flows correspond to a minimum of energy dissipation

Reversibility implies symmetry



Reversibility implies symmetry



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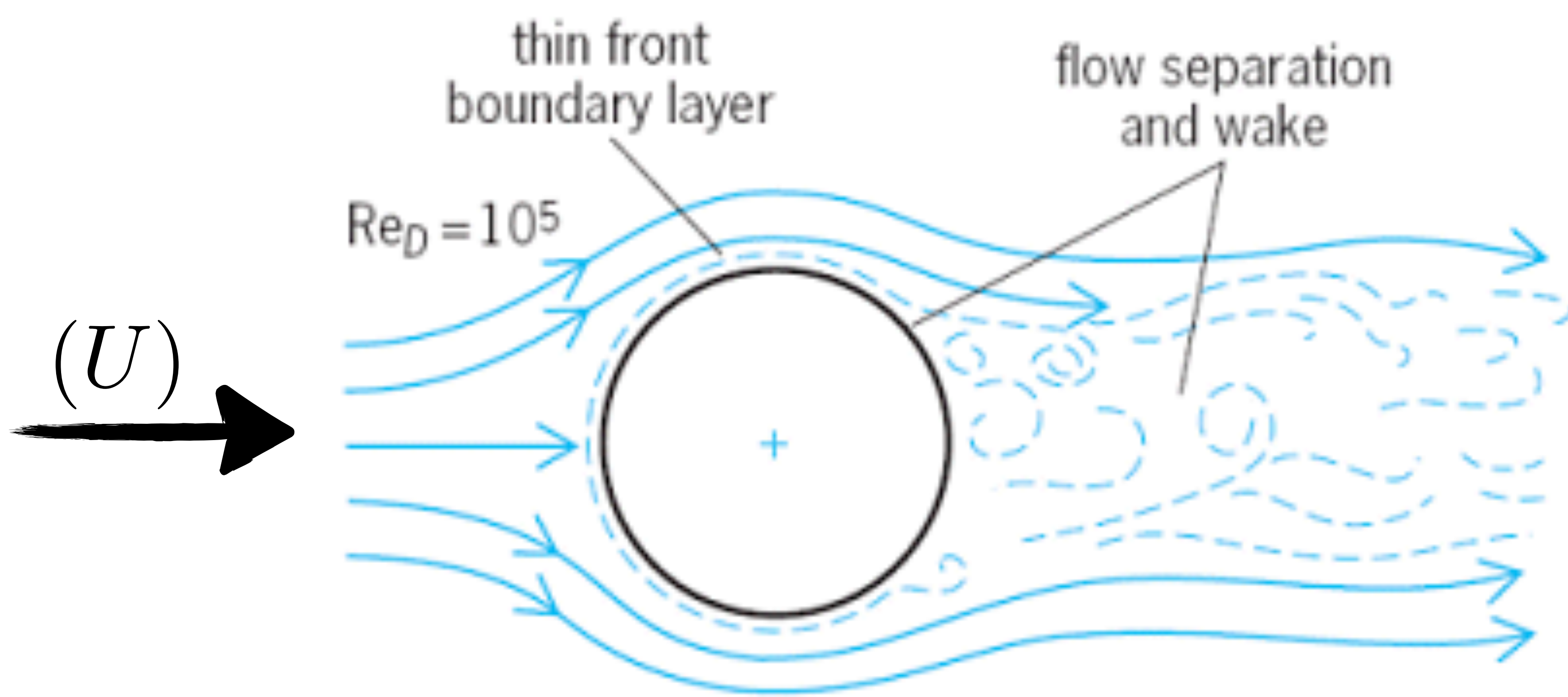
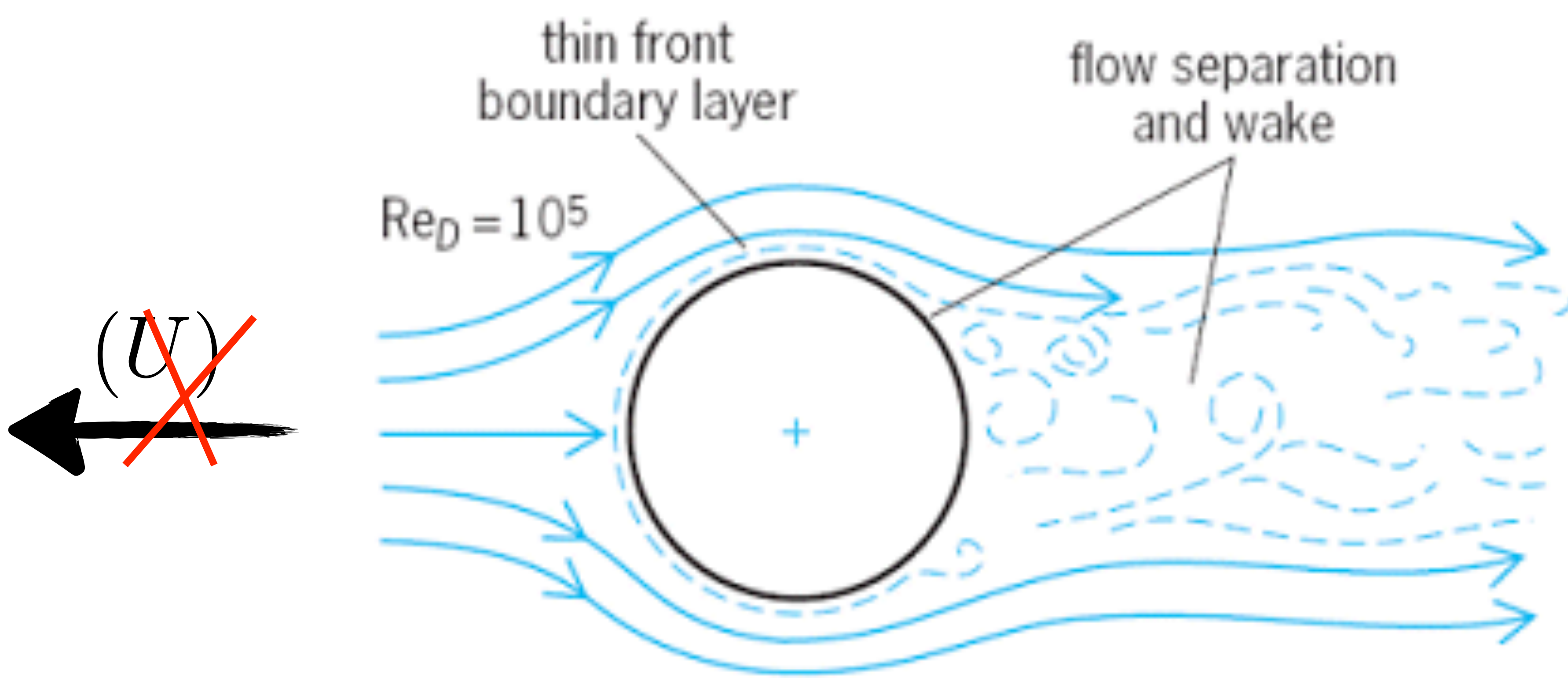
Reversibility implies symmetry

This flow is not the same as its reverse



Reversibility implies symmetry

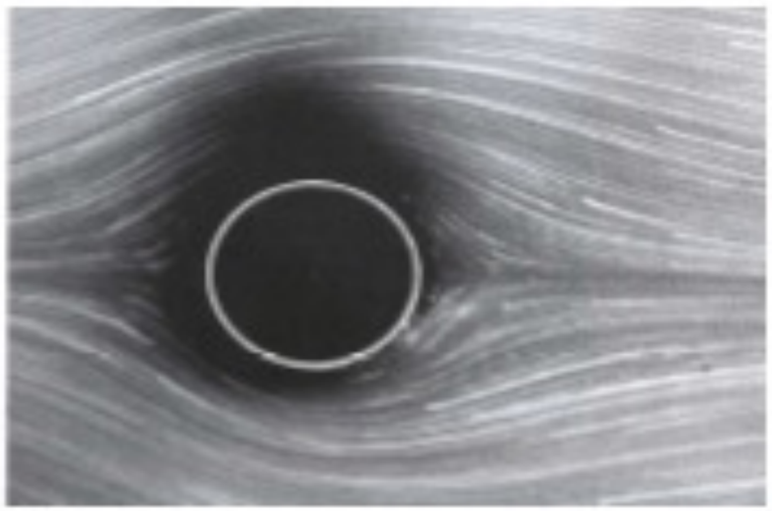
This flow is not the same as its reverse



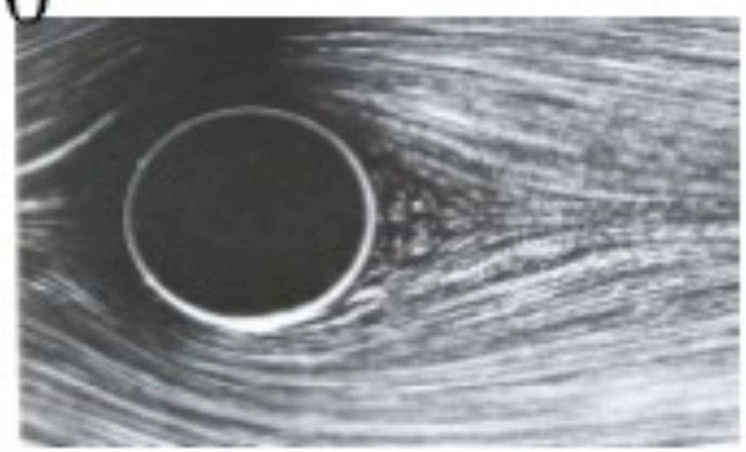
This flow cannot be solution to Stokes equation

Increasing complexity of flows with increasing Reynolds number: flow past a circular cylinder

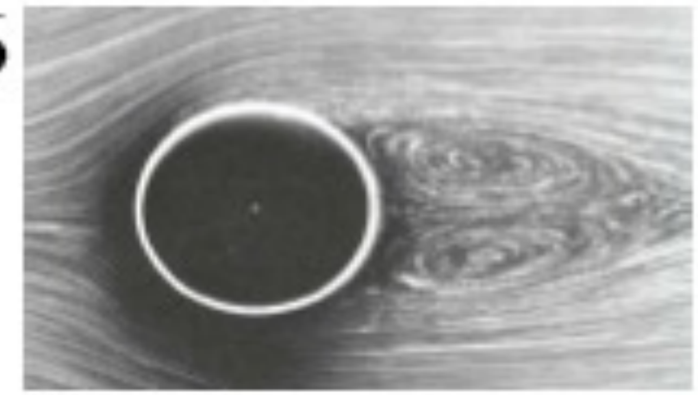
Re=1



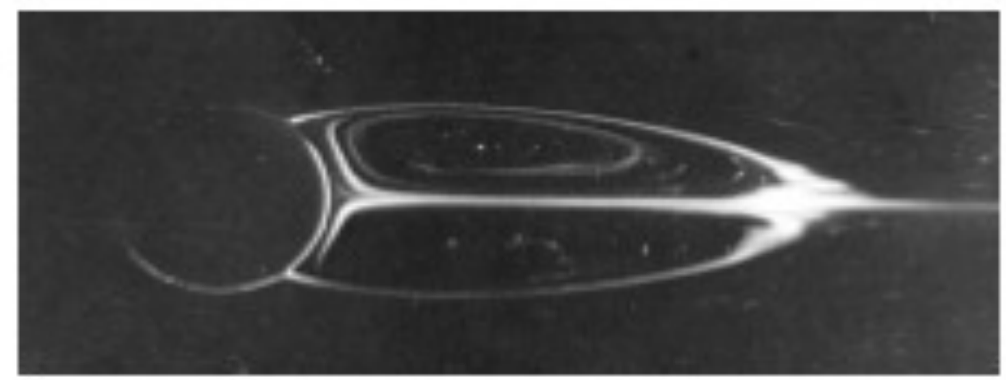
Re=10



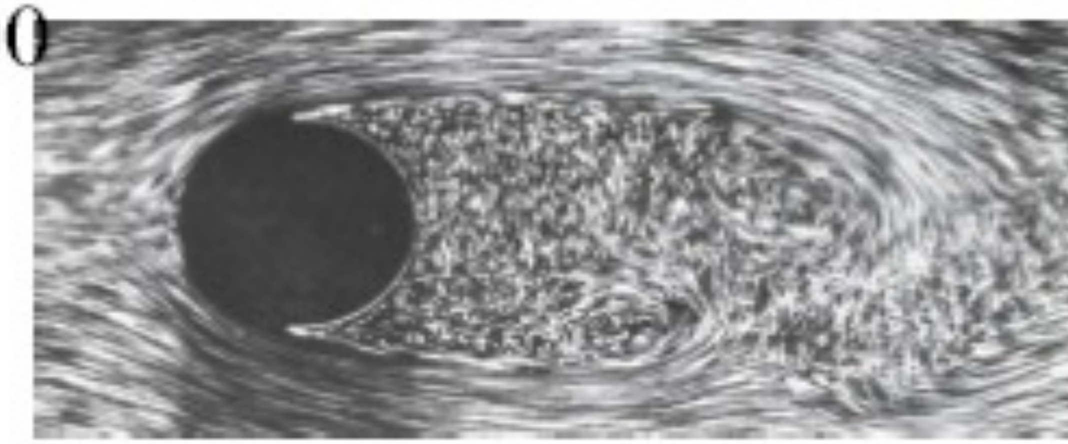
Re=26



Re=41

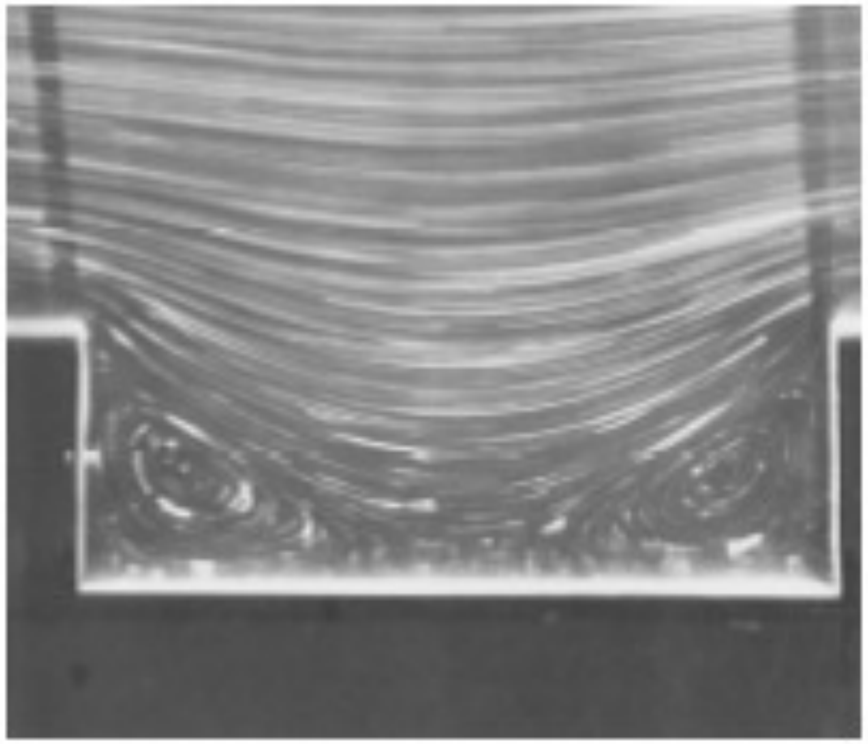


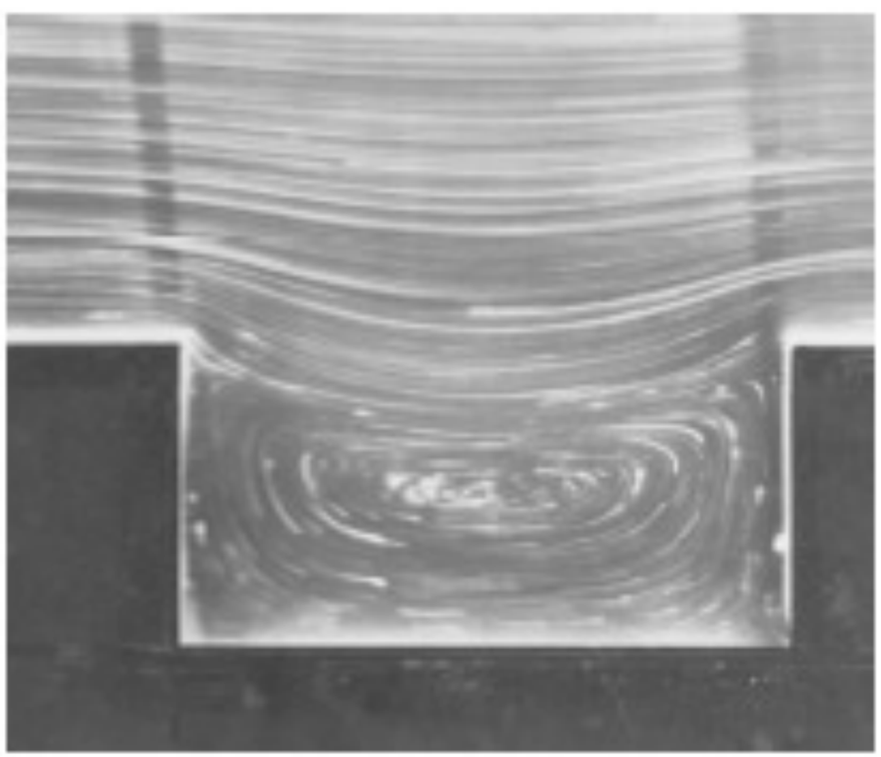
Re=2000



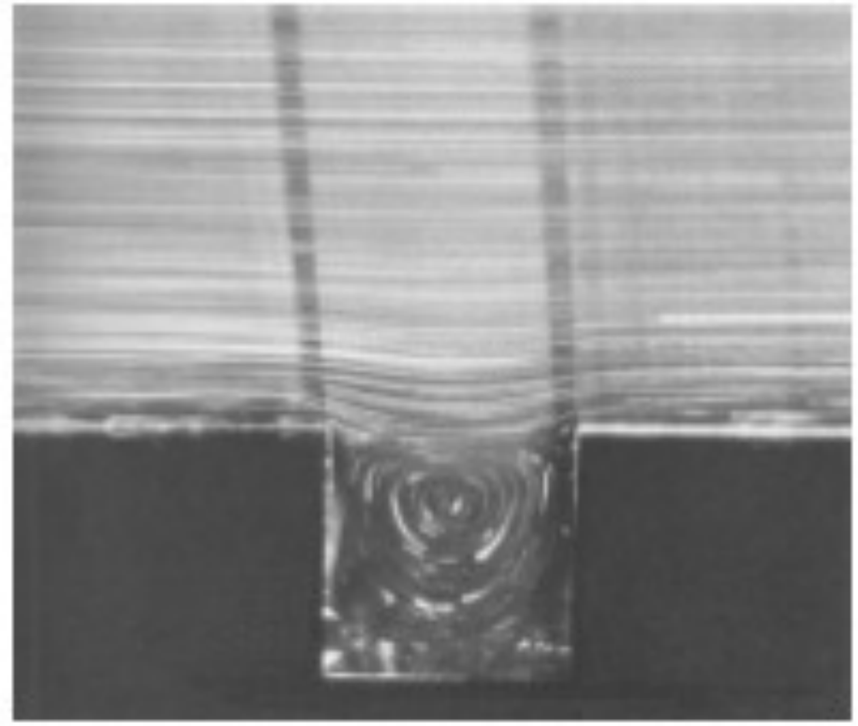
Reference: Van Dyke, *Album of Fluid Motion*

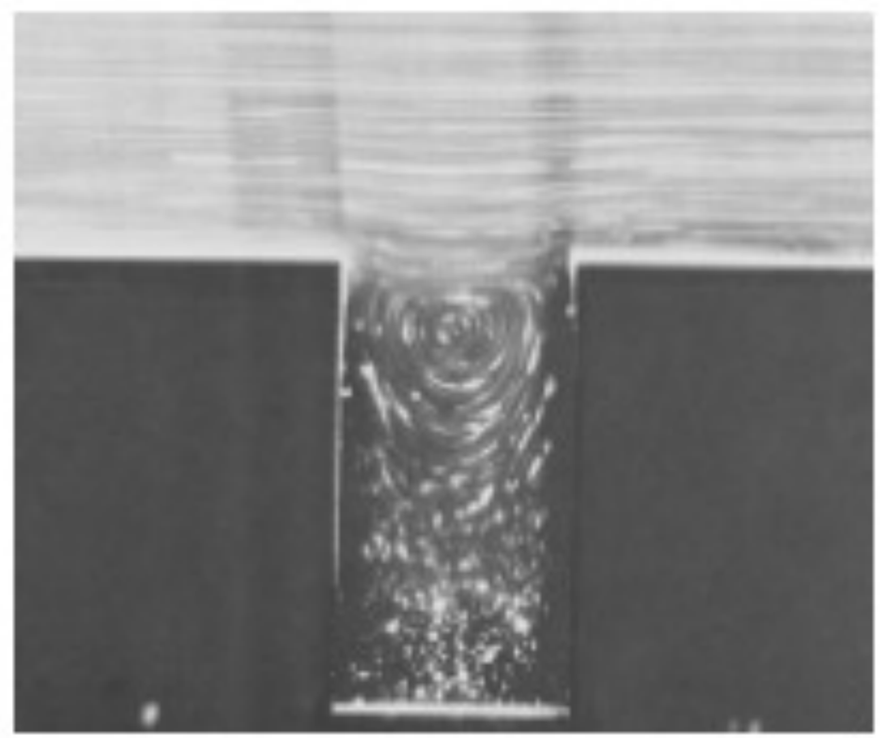
Classical solutions of Stokes Flows



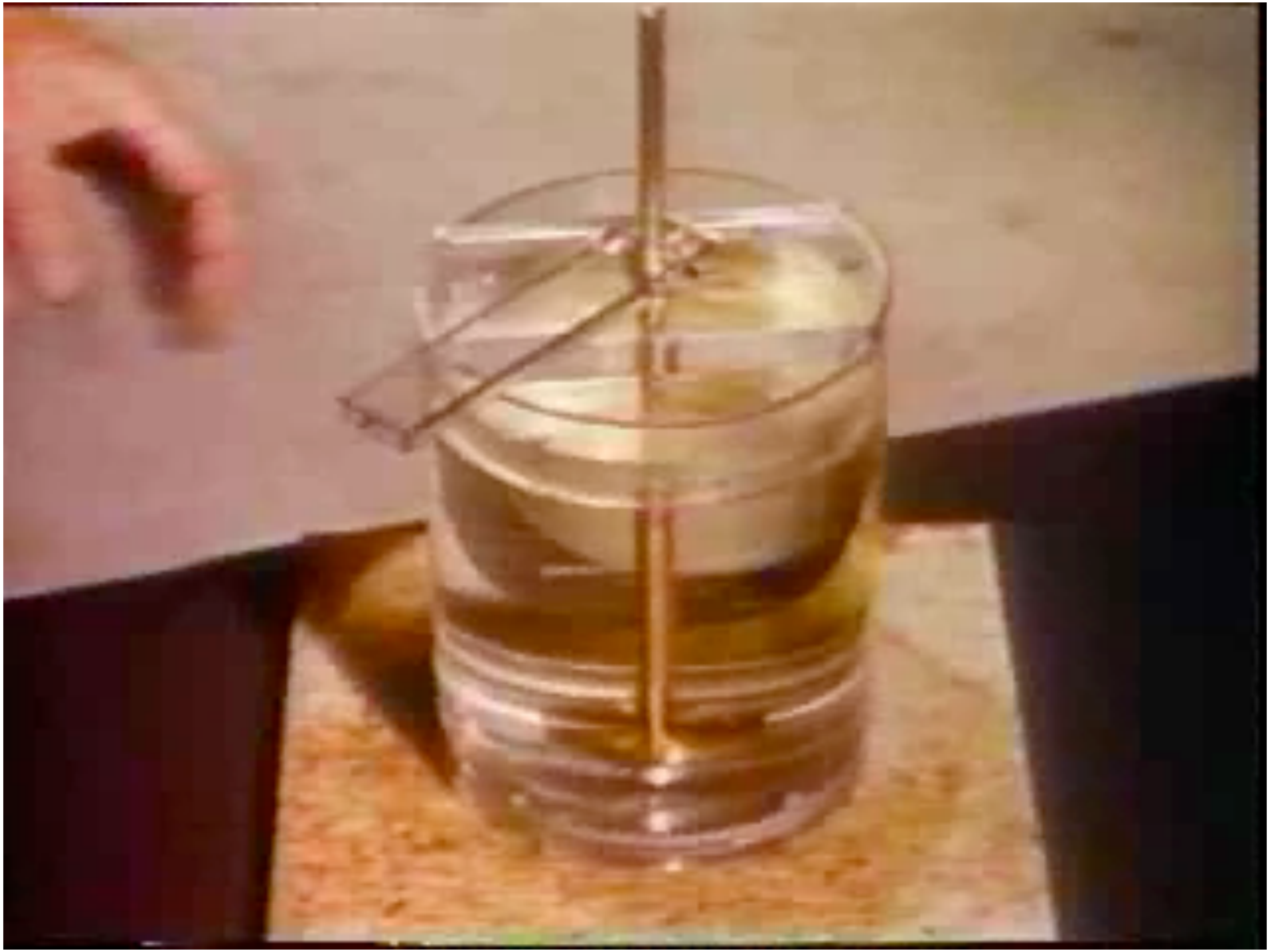


Systematically change the aspect ratio of the cavity: note appearance of one or more eddies





G.I. Taylor demonstrates reversibility



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Interpretation of the Reynolds number I

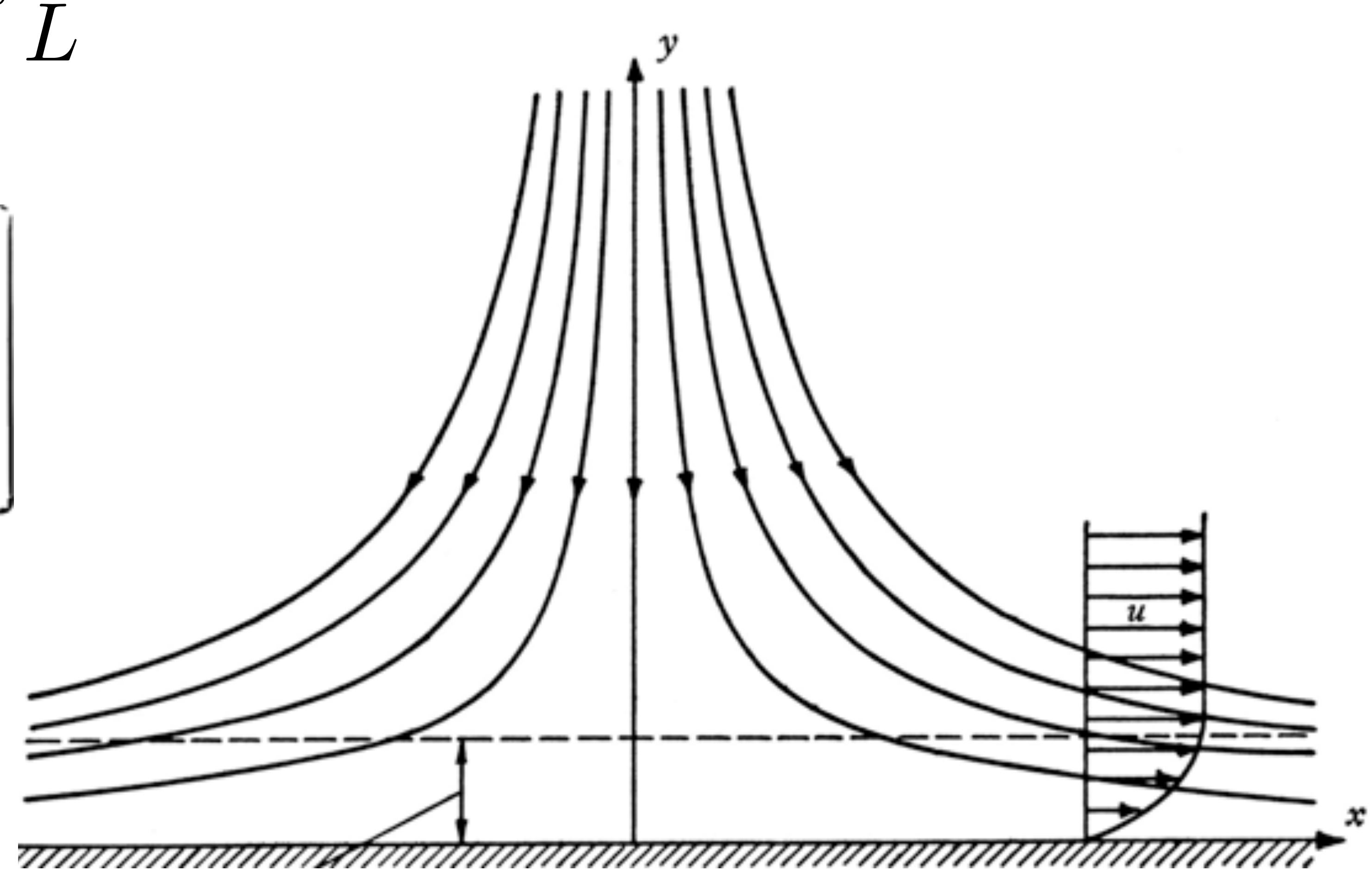
- Stagnation pressure on a solid

Inertial stresses $$p \sim \rho U^2$$

- Shear stress due to the velocity gradient

Viscous stresses $$\sigma \sim \mu \frac{U}{L}$$

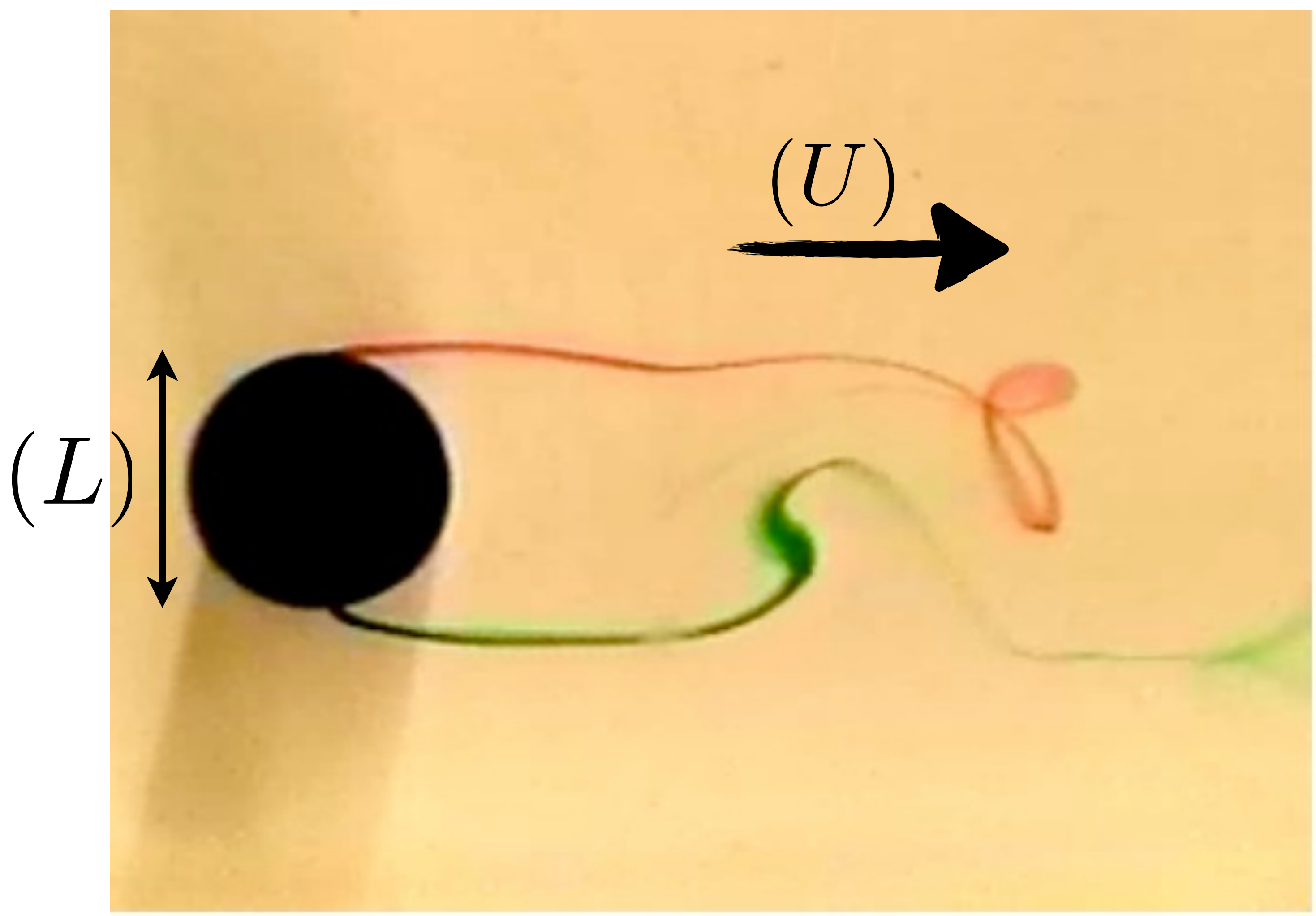
$$\boxed{\frac{p}{\sigma} \sim \frac{\rho U^2}{\mu U/L} \sim \frac{\rho U L}{\mu} = Re}$$



Interpretation of the Reynolds number II

- Advection time

$$\tau_{adv} \sim \frac{L}{U}$$



Interpretation of the Reynolds number II

Change in boundary velocity transmitted *via* a diffusive process

- Viscous diffusion time

$$\tau_{diff} \sim \frac{L^2 \rho}{\mu} = \frac{L^2}{\nu}$$

$$[\nu] = \frac{L}{T^2}$$

Kinematic viscosity

Boundary moving below a high-viscosity oil

Interpretation of the Reynolds number II

- Viscous diffusion time

$$\tau_{diff} \sim \frac{L^2 \rho}{\mu} = \frac{L^2}{\nu} \qquad [\nu] = \frac{L}{T^2}$$

- Advection time

$$\tau_{adv} \sim \frac{L}{U}$$

$$\boxed{\frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L} \sim \frac{UL}{\nu} = Re}$$

Back to N-S. to Stokes equation

$$\rho \left[\frac{\partial \vec{u}}{\partial t} + \vec{u} \cdot \vec{\nabla} \vec{u} \right] = -\vec{\nabla} p + \mu \vec{\nabla}^2 \vec{u}$$

Negligible if $\frac{p}{\sigma} \sim \frac{\rho U^2}{\mu U / L}$ << 1

(Viscous dominate over inertial stresses)

Negligible if $\frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L}$ << 1

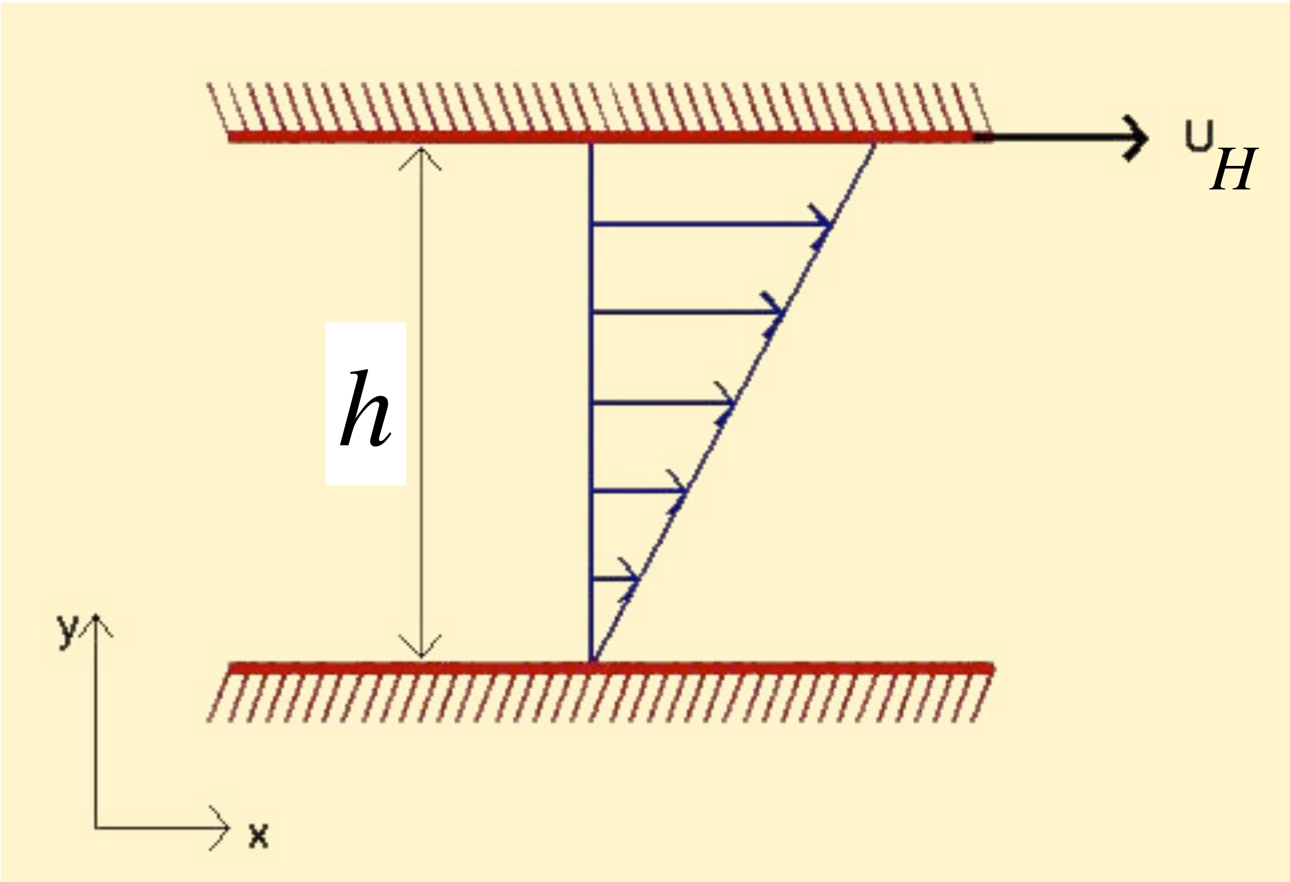
(fluid adjusts immediately to variations in BC)

Two basic flows

- Boundary driven flow in a gap
  (Couette flow)

- Pressure driven flow in a tube
  (Poiseuille flow)

Couette flow



No pressure term.
Stokes equation:

$$\mu \frac{\partial^2 u}{\partial y^2} = 0$$

$$u(y=0) = 0 \quad u(y=h) = u_H$$

$$u = u_H \left(\frac{y}{h}\right)$$

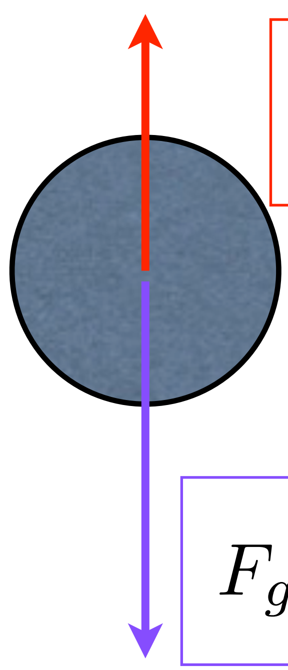
Shear stress on the wall:

$$\tau_{xy} = \mu \frac{\partial u}{\partial y} = \mu \frac{u_H}{h}$$

For a plate of area *A*

$$F = A \cdot \tau_{xy}$$

Force on a falling sphere



$$\boxed{F_d = 6\pi\mu R U}$$

$$\boxed{F_g = (\rho_s - \rho_l) g \frac{4}{3}\pi R^3}$$

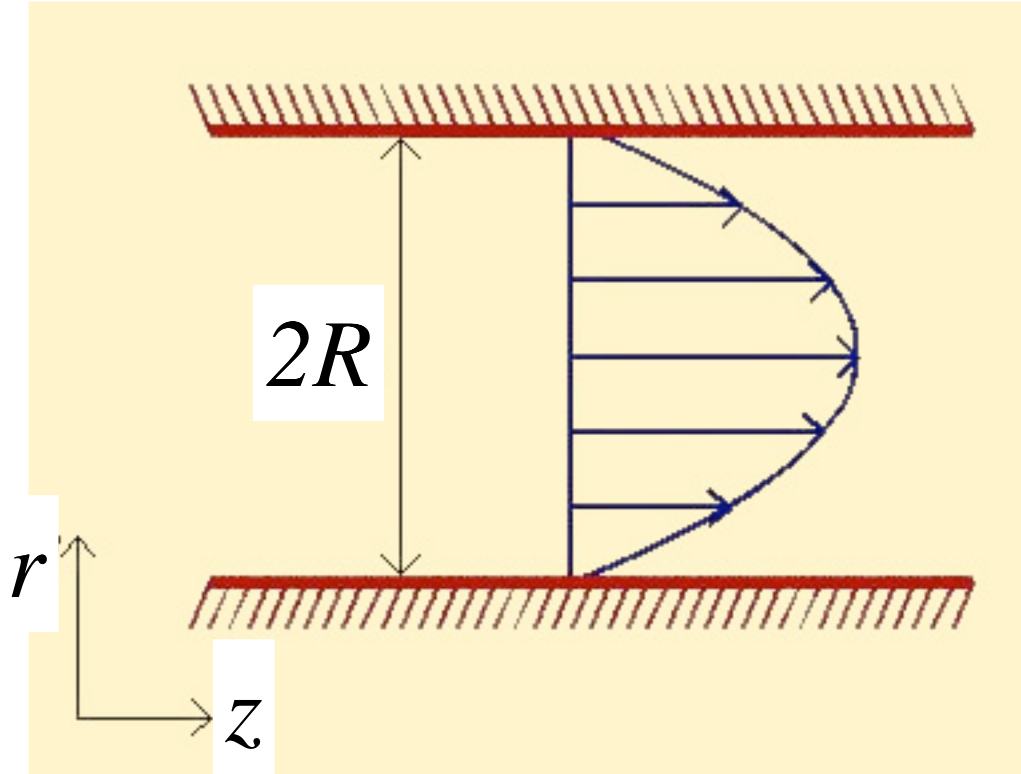
Settling velocity

$$U = \frac{2}{9}\frac{g\Delta\rho}{\mu}R^2$$

Verify drag force scaling

Left sphere twice as big as right sphere

Poiseuille flow



Assume constant pressure gradient

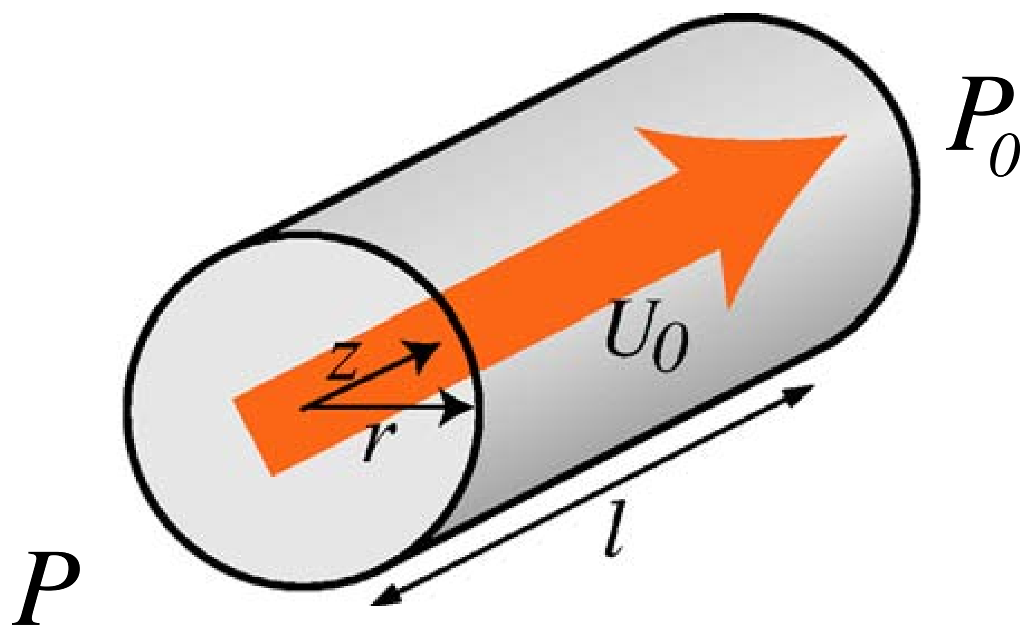
$$\nabla p = \frac{\partial p}{\partial z} = cst$$

Velocity inside cylindrical tube of radius R

$$\boxed{u_z = \frac{-1}{4\mu}\frac{\partial p}{\partial z}(R^2 - r^2)}$$

Flow rate: $$Q = -\frac{\pi R^4}{8\mu}\frac{\partial p}{\partial z}$$

Tube of length *l*



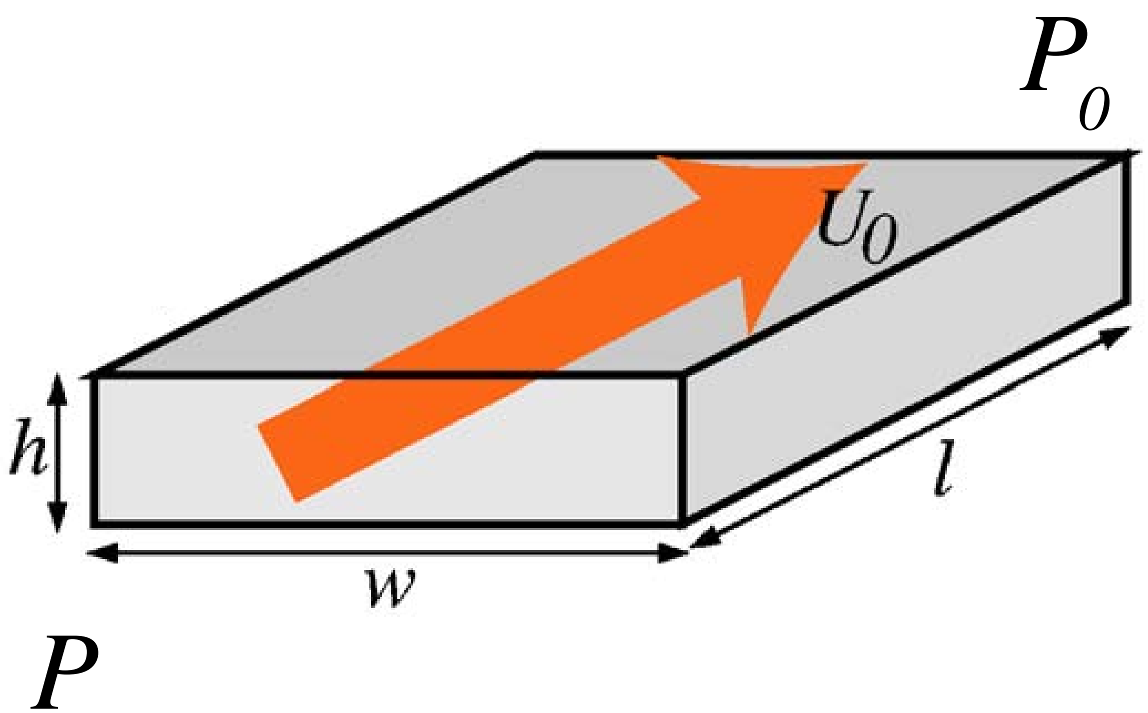
$$Q = \frac{\pi R^4}{8\mu L}\Delta p = \frac{\Delta p}{\mathcal{R}}$$

Hydrodynamic resistance

$$\boxed{\mathcal{R} = \frac{8\mu L}{\pi R^4}}$$

You don't need to solve the fluids equations to know the flow rate

Microchannel



$$Q \simeq \frac{wh^3}{12\mu L}\left[1 - 6\left(\frac{2}{\pi}\right)^5 \frac{h}{w}\right]\Delta p$$

Hydrodynamic resistance

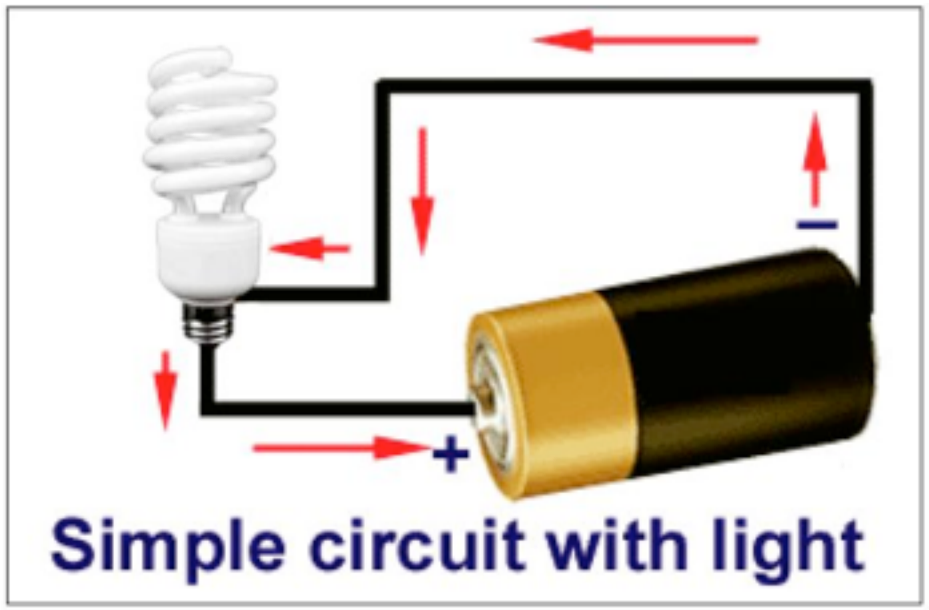
$$\mathcal{R} = \frac{12\mu L}{wh^3}\left[1 - 6\left(\frac{2}{\pi}\right)^5 \frac{h}{w}\right]^{-1}$$

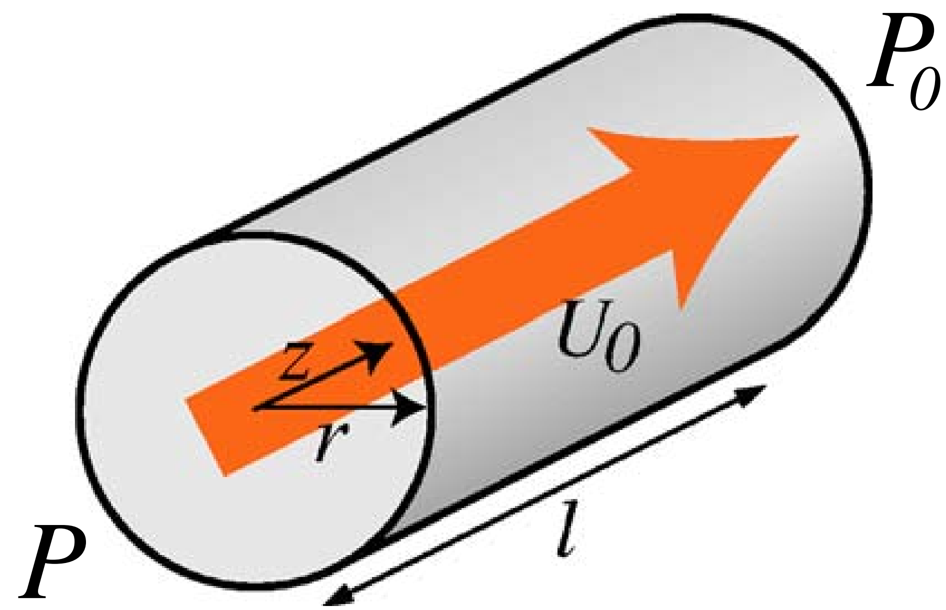
If $h << w$

$$\mathcal{R} = \frac{12\mu L}{wh^3}$$

Strong dependence on h

Analogy with electrical circuits

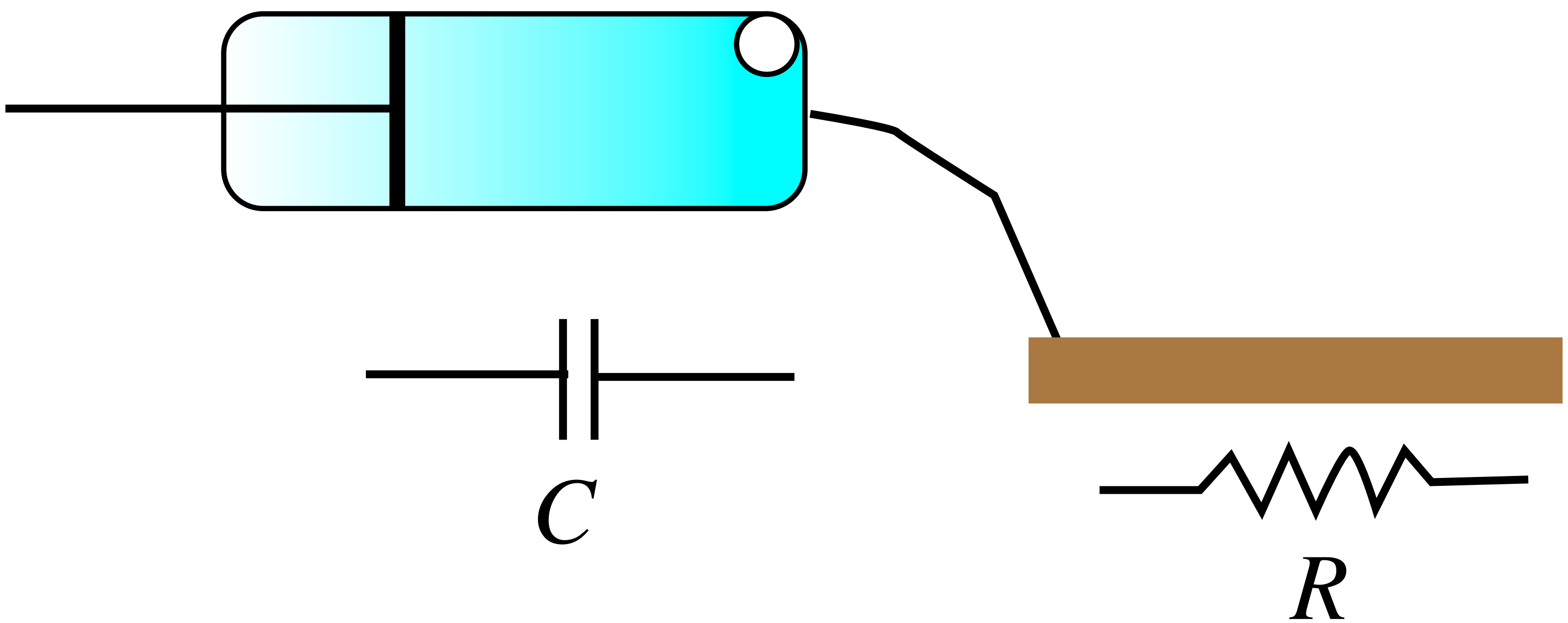




Electrical	**Fluidic**
Voltage drop (ΔV)	Pressure drop (Δp)
Current (I)	Flow rate (Q)
Resistance	Fluidic resistance ($\mathcal{R}$)
Capacitance ($\mathcal{C}$)	Mechanical compliance ($\mathcal{K}$)
Inductance ($\mathcal{L}$)	~~Inertia~~

Compliance

Compressibility is a way to store pressure:
Fluidic capacitance



Compliance associated with an air bubble:

$$C_h = \frac{p_0 V_0}{p^2} \longrightarrow \frac{\partial P}{\partial t} = \frac{1}{C_h} Q$$

R-C circuit

Hydrodynamic resistance

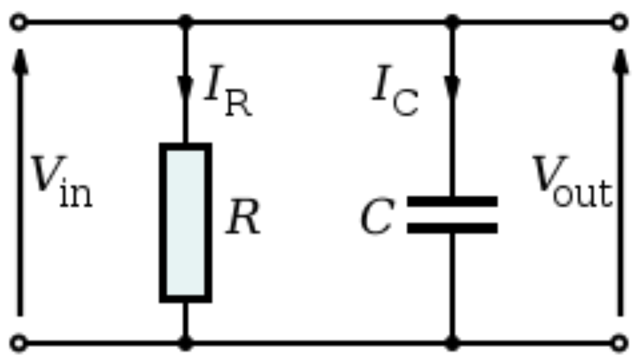
$$P = \mathcal{R}Q$$

Compliance

$$\frac{\partial P}{\partial t} = \frac{1}{C_h}Q$$

Therefore:

$$\frac{\partial P}{\partial t} = \frac{1}{\mathcal{R}C_h}P$$



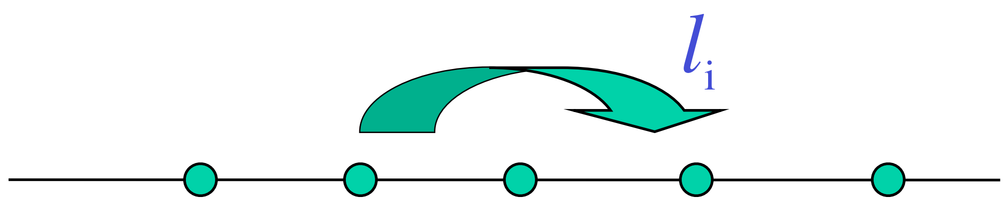
$$P(t) \sim e^{-t/\mathcal{R}C_h}$$

Large resistance and large compliance imply that the system will respond very slowly

A few words about diffusion

A few words about diffusion

Molecular scale model:



A molecule performs a random walk with a certain step size during every time step.

Mean field model:

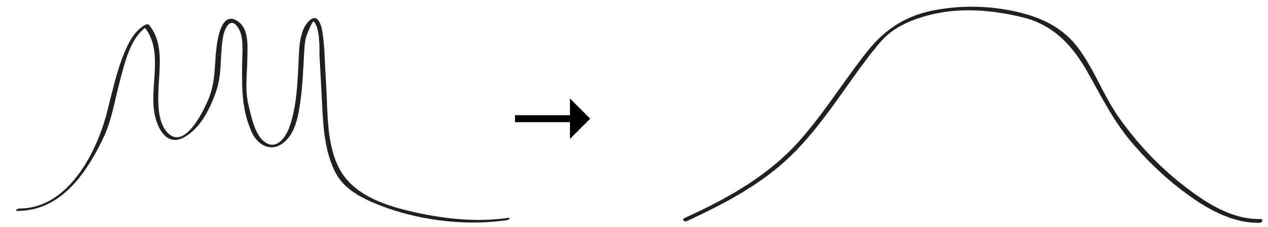
A chemical species is transported «down» the concentration gradient

$$\boxed{\frac{\partial C(x,t)}{\partial t} = D\frac{\partial^2 C(x,t)}{\partial x^2}}$$

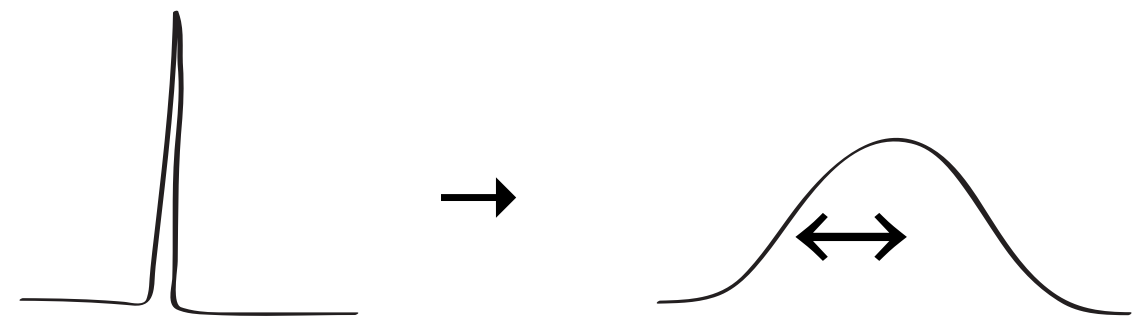
Useful solutions

Diffusion quickly evens out short-wave variations

$$\frac{\partial^2 C}{\partial x^2} \gg 1$$

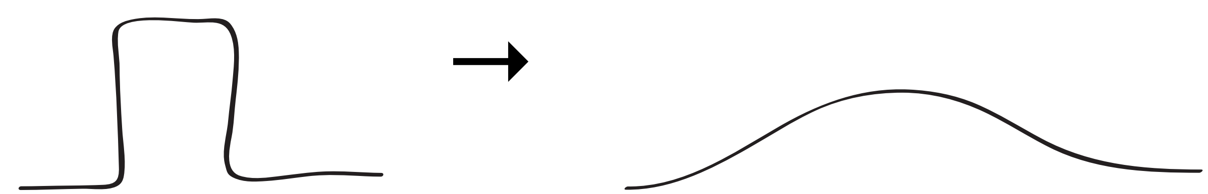


Delta-function → Gaussian

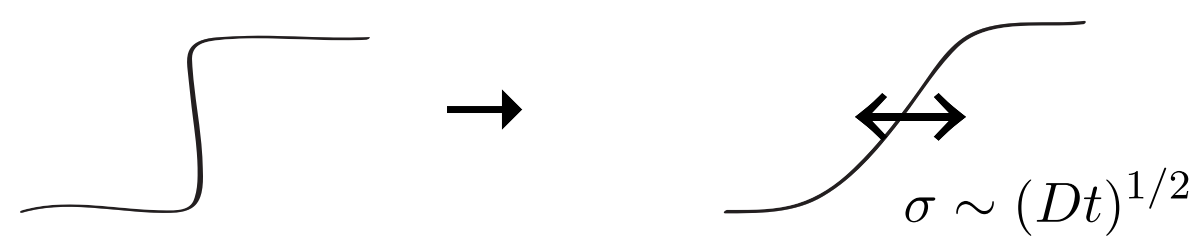


$\sigma \sim (Dt)^{1/2}$

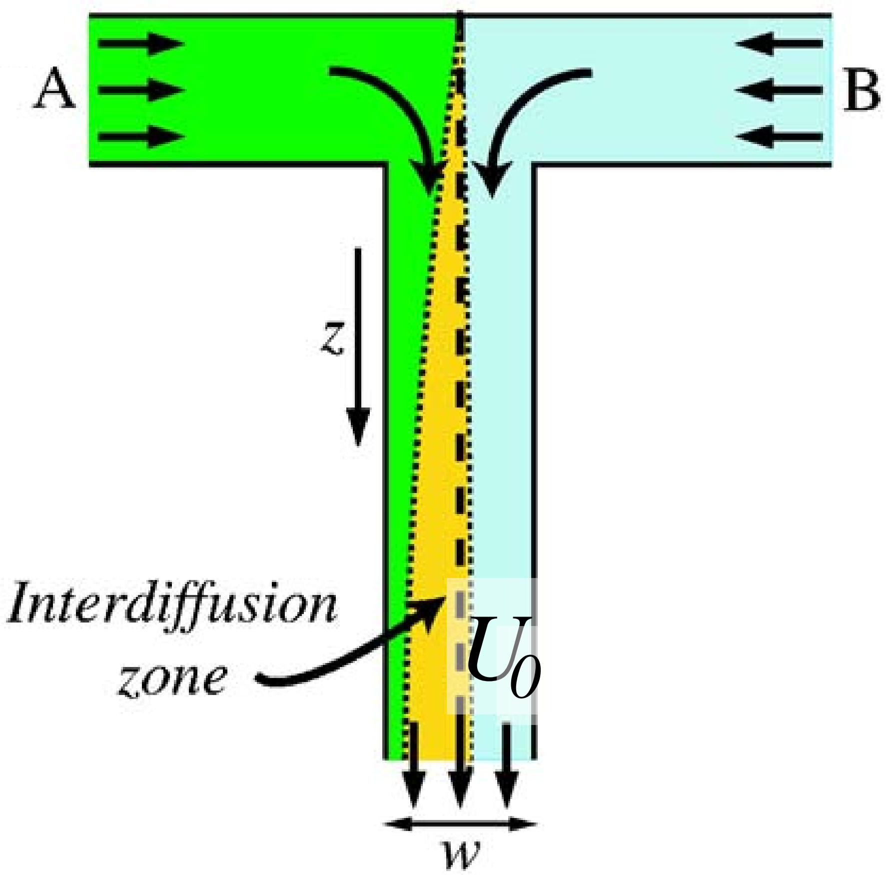
Top hat → Gaussian



Heaviside-function → error function



Diffusion in a T-channel



How far will the species diffuse?

Time to diffuse whole width

$$\tau_{diff} \sim \frac{w^2}{D}$$

Distance travelled during this time

$$Z \sim U_0 \tau_{diff} \sim U_0 \frac{w^2}{D}$$

How many channel widths?

$$\boxed{\frac{Z}{w} \sim \frac{U_0 w}{D} = Pe}$$

Peclet number as ratio of two lengths

Pe vs. Re

$$\boxed{\frac{Z}{w} \sim \frac{U_0 w}{D} = Pe}$$

$$\boxed{\frac{\tau_{diff}}{\tau_{adv}} \sim \frac{L^2}{\nu} \cdot \frac{U}{L} \sim \frac{UL}{\nu} = Re}$$

Reynolds number as a
Peclet number for
momentum transfer